

to the Rev. Dr. Peacock, M. A.
ON from the Author.

A GENERAL METHOD

IN

D Y N A M I C S.

BY
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XV. *On a General Method in Dynamics; by which the Study of the Motions of all free Systems of attracting or repelling Points is reduced to the Search and Differentiation of one central Relation, or characteristic Function.* By WILLIAM ROWAN HAMILTON, Member of several scientific Societies in the British Dominions, and of the American Academy of Arts and Sciences, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland. Communicated by Captain BEAUFORT, R.N. F.R.S.

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Introductory Remarks.

THE theoretical development of the laws of motion of bodies is a problem of such interest and importance, that it has engaged the attention of all the most eminent mathematicians, since the invention of dynamics as a mathematical science by GALILEO, and especially since the wonderful extension which was given to that science by NEWTON. Among the successors of those illustrious men, LAGRANGE has perhaps done more than any other analyst, to give extent and harmony to such deductive researches, by showing that the most varied consequences respecting the motions of systems of bodies may be derived from one radical formula; the beauty of the method so suiting the dignity of the results, as to make of his great work a kind of scientific poem. But the science of force, or of power acting by law in space and time, has undergone already another revolution, and has become already more dynamic, by having almost dismissed the conceptions of solidity and cohesion, and those other material ties, or geometrically imaginable conditions, which LAGRANGE so happily reasoned on, and by tending more and more to resolve all connexions and actions of bodies into attractions and repulsions of points: and while the science is advancing thus in one direction by the improvement of physical views, it may advance in another direction also by the invention of mathematical methods. And the method proposed in the present essay, for the deductive study of the motions of attracting or repelling systems, will perhaps be received with indulgence, as an attempt to assist in carrying forward so high an inquiry.

In the methods commonly employed, the determination of the motion of a free point in space, under the influence of accelerating forces, depends on the integration of three equations in ordinary differentials of the second order; and the determination of the motions of a system of free points, attracting or repelling one another, depends on the integration of a system of such equations, in number threefold the

number of the attracting or repelling points, unless we previously diminish by unity this latter number, by considering only relative motions. Thus, in the solar system, when we consider only the mutual attractions of the sun and of the ten known planets, the determination of the motions of the latter about the former is reduced, by the usual methods, to the integration of a system of thirty ordinary differential equations of the second order, between the coordinates and the time; or, by a transformation of LAGRANGE, to the integration of a system of sixty ordinary differential equations of the first order, between the time and the elliptic elements: by which integrations, the thirty varying coordinates, or the sixty varying elements, are to be found as functions of the time. In the method of the present essay, this problem is reduced to the search and differentiation of a single function, which satisfies two partial differential equations of the first order and of the second degree: and every other dynamical problem, respecting the motions of any system, however numerous, of attracting or repelling points, (even if we suppose those points restricted by any conditions of connexion consistent with the law of living force,) is reduced, in like manner, to the study of one central function, of which the form marks out and characterizes the properties of the moving system, and is to be determined by a pair of partial differential equations of the first order, combined with some simple considerations. The difficulty is therefore at least transferred from the integration of many equations of one class to the integration of two of another: and even if it should be thought that no practical facility is gained, yet an intellectual pleasure may result from the reduction of the most complex and, probably, of all researches respecting the forces and motions of body, to the study of one characteristic function*, the unfolding of one central relation.

The present essay does not pretend to treat fully of this extensive subject,—a task which may require the labours of many years and many minds; but only to suggest the thought and propose the path to others. Although, therefore, the method may be used in the most varied dynamical researches, it is at present only applied to the orbits and perturbations of a system with any laws of attraction or repulsion, and with one predominant mass or centre of predominant energy; and only so far, even in this one research, as appears sufficient to make the principle itself understood. It may be mentioned here, that this dynamical principle is only another form of that idea which has already been applied to optics in the *Theory of systems of rays*, and that an intention of applying it to the motions of systems of bodies was announced †.

* LAGRANGE and, after him, LAPLACE and others, have employed a single function to express the different forces of a system, and so to form in an elegant manner the differential equations of its motion. By this conception, great simplicity has been given to the statement of the problem of dynamics; but the solution of that problem, or the expression of the motions themselves, and of their integrals, depends on a very different and hitherto unimagined function, as it is the purpose of this essay to show.

† Transactions of the Royal Irish Academy, vol. xv. page 80. A notice of this dynamical principle was also lately given in an article "On a general Method of expressing the Paths of Light and of the Planets," published in the Dublin University Review for October 1833.

at the publication of that theory. And besides the idea itself, the manner of calculation also, which has been thus exemplified in the sciences of optics and dynamics, seems not confined to those two sciences, but capable of other applications; and the peculiar combination which it involves, of the principles of variations with those of partial differentials, for the determination and use of an important class of integrals, may constitute, when it shall be matured by the future labours of mathematicians, a separate branch of analysis.

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Integration of the Equations of Motion of a System, characteristic Function of such Motion, and Law of varying Action.

1. The known differential equations of motion of a system of free points, repelling or attracting one another according to any functions of their distances, and not disturbed by any foreign force, may be comprised in the following formula:

$$\Sigma \cdot m (x'' \delta x + y'' \delta y + z'' \delta z) = \delta U. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (1.)$$

In this formula the sign of summation Σ extends to all the points of the system; m is, for any one such point, the constant called its mass; x'', y'', z'' , are its component accelerations, or the second differential coefficients of its rectangular coordinates x, y, z , taken with respect to the time; $\delta x, \delta y, \delta z$, are any arbitrary infinitesimal displacements which the point can be imagined to receive in the same three rectangular directions; and δU is the infinitesimal variation corresponding, of a function U of the masses and mutual distances of the several points of the system, of which the form depends on the laws of their mutual actions, by the equation

$$U = \Sigma \cdot m m_i f(r), \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (2.)$$

r being the distance between any two points m, m_i , and the function $f(r)$ being such that its derivative or differential coefficient $f'(r)$ expresses the law of their repulsion, being negative in the case of attraction. The function which has been here called U , may be named the *force-function* of a system: it is of great utility in theoretical mechanics, into which it was introduced by LAGRANGE, and it furnishes the following elegant forms for the differential equations of motion, included in the formula (1.):

$$\left. \begin{array}{l} m_1 x''_1 = \frac{\delta U}{\delta x_1}; \quad m_2 x''_2 = \frac{\delta U}{\delta x_2}; \quad \dots \quad m_n x''_n = \frac{\delta U}{\delta x_n}; \\ m_1 y''_1 = \frac{\delta U}{\delta y_1}; \quad m_2 y''_2 = \frac{\delta U}{\delta y_2}; \quad \dots \quad m_n y''_n = \frac{\delta U}{\delta y_n}; \\ m_1 z''_1 = \frac{\delta U}{\delta z_1}; \quad m_2 z''_2 = \frac{\delta U}{\delta z_2}; \quad \dots \quad m_n z''_n = \frac{\delta U}{\delta z_n}; \end{array} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.)$$

the second members of these equations being the partial differential coefficients of

the first order of the function U . But notwithstanding the elegance and simplicity of this known manner of stating the principal problem of dynamics, the difficulty of solving that problem, or even of expressing its solution, has hitherto appeared insuperable; so that only seven intermediate integrals, or integrals of the first order, with as many arbitrary constants, have hitherto been found for these general equations of motion of a system of n points, instead of $3n$ intermediate and $3n$ final integrals, involving ultimately $6n$ constants; nor has any integral been found which does not need to be integrated again. No general solution has been obtained assigning (as a complete solution ought to do) $3n$ relations between the n masses m_1, m_2, \dots, m_n , the $3n$ varying coordinates $x_1, y_1, z_1, \dots, x_n, y_n, z_n$, the varying time t , and the $6n$ initial data of the problem, namely, the initial coordinates $a_1, b_1, c_1, \dots, a_n, b_n, c_n$, and their initial rates of increase, $a'_1, b'_1, c'_1, \dots, a'_n, b'_n, c'_n$; the quantities called here initial being those which correspond to the arbitrary origin of time. It is, however, possible (as we shall see) to express these long-sought relations by the partial differential coefficients of a new central or radical function, to the search and employment of which the difficulty of mathematical dynamics becomes henceforth reduced.

2. If we put for abridgement

$$T = \frac{1}{2} \sum m (x'^2 + y'^2 + z'^2), \quad \dots \dots \dots \dots \dots \quad (4.)$$

so that $2T$ denotes, as in the Mécanique Analytique, the whole living force of the system; (x', y', z' , being here, according to the analogy of our foregoing notation, the rectangular components of velocity of the point m , or the first differential coefficients of its coordinates taken with respect to the time;) an easy and well known combination of the differential equations of motion, obtained by changing in the formula (1.) the variations to the differentials of the coordinates, may be expressed in the following manner,

$$dT = dU, \quad \dots \dots \dots \dots \dots \quad (5.)$$

and gives, by integration, the celebrated law of living force, under the form

$$T = U + H. \quad \dots \dots \dots \dots \dots \quad (6.)$$

In this expression, which is one of the seven known integrals already mentioned, the quantity H is independent of the time, and does not alter in the passage of the points of the system from one set of positions to another. We have, for example, an initial equation of the same form, corresponding to the origin of time, which may be written thus,

$$T_0 = U_0 + H. \quad \dots \dots \dots \dots \dots \quad (7.)$$

The quantity H may, however, receive any arbitrary increment whatever, when we pass in thought from a system moving in one way, to the same system moving in another, with the same dynamical relations between the accelerations and positions of its points, but with different initial data; but the increment of H , thus obtained,

is evidently connected with the analogous increments of the functions T and U , by the relation

$$\Delta T = \Delta U + \Delta H, \dots \dots \dots \dots \dots \dots \quad (8.)$$

which, for the case of infinitesimal variations, may conveniently be written thus,

$$\delta T = \delta U + \delta H; \dots \dots \dots \dots \dots \dots \quad (9.)$$

and this last relation, when multiplied by dt , and integrated, conducts to an important result. For it thus becomes, by (4.) and (1.),

$$\int \Sigma . m (dx \cdot \delta x' + dy \cdot \delta y' + dz \cdot \delta z') =$$

$$\int \Sigma . m (dx' \cdot \delta x + dy' \cdot \delta y + dz' \cdot \delta z) + \int \delta H \cdot dt, \dots \dots \dots \quad (10.)$$

that is, by the principles of the calculus of variations,

$$\delta V = \Sigma . m (x' \delta x + y' \delta y + z' \delta z) - \Sigma . m (a' \delta a + b' \delta b + c' \delta c) + t \delta H, \dots \quad (A.)$$

if we denote by V the integral

$$V = \int \Sigma . m (x' dx + y' dy + z' dz) = \int_0^t 2T dt, \dots \dots \dots \quad (B.)$$

namely, the accumulated living force, called often the action of the system, from its initial to its final position.

If, then, we consider (as it is easy to see that we may) the action V as a function of the initial and final coordinates, and of the quantity H , we shall have, by (A.), the following groups of equations; first, the group,

$$\left. \begin{array}{l} \frac{\delta V}{\delta x_1} = m_1 x'_1; \frac{\delta V}{\delta x_2} = m_2 x'_2; \dots \frac{\delta V}{\delta x_n} = m_n x'_n; \\ \frac{\delta V}{\delta y_1} = m_1 y'_1; \frac{\delta V}{\delta y_2} = m_2 y'_2; \dots \frac{\delta V}{\delta y_n} = m_n y'_n; \\ \frac{\delta V}{\delta z_1} = m_1 z'_1; \frac{\delta V}{\delta z_2} = m_2 z'_2; \dots \frac{\delta V}{\delta z_n} = m_n z'_n; \end{array} \right\} \dots \dots \dots \quad (C.)$$

Secondly, the group,

$$\left. \begin{array}{l} \frac{\delta V}{\delta a_1} = -m_1 a'_1; \frac{\delta V}{\delta a_2} = -m_2 a'_2; \dots \frac{\delta V}{\delta a_n} = -m_n a'_n; \\ \frac{\delta V}{\delta b_1} = -m_1 b'_1; \frac{\delta V}{\delta b_2} = -m_2 b'_2; \dots \frac{\delta V}{\delta b_n} = -m_n b'_n; \\ \frac{\delta V}{\delta c_1} = -m_1 c'_1; \frac{\delta V}{\delta c_2} = -m_2 c'_2; \dots \frac{\delta V}{\delta c_n} = -m_n c'_n; \end{array} \right\} \dots \dots \dots \quad (D.)$$

and finally, the equation,

$$\frac{\delta V}{\delta H} = t; \dots \quad (E.)$$

So that if this function V were known, it would only remain to eliminate H between the $3n+1$ equations (C.) and (E.), in order to obtain all the $3n$ intermediate integrals, or between (D.) and (E.) to obtain all the $3n$ final integrals of the differential equations of motion; that is, ultimately, to obtain the $3n$ sought relations between

the $3n$ varying coordinates and the time, involving also the masses and the $6n$ initial data above mentioned ; the discovery of which relations would be (as we have said) the general solution of the general problem of dynamics. We have, therefore, at least reduced that general problem to the search and differentiation of a single function V , which we shall call on this account the **CHARACTERISTIC FUNCTION** of motion of a system ; and the equation (A.), expressing the fundamental law of its variation, we shall call the *equation of the characteristic function*, or the **LAW OF VARYING ACTION**.

3. To show more clearly that the action or accumulated living force of a system, or in other words, the integral of the product of the living force by the element of the time, may be regarded as a function of the $6n+1$ quantities already mentioned, namely, of the initial and final coordinates, and of the quantity H , we may observe, that whatever depends on the manner and time of motion of the system may be considered as such a function ; because the initial form of the law of living force, when combined with the $3n$ known or unknown relations between the time, the initial data, and the varying coordinates, will always furnish $3n+1$ relations, known or unknown, to connect the time and the initial components of velocities with the initial and final coordinates, and with H . Yet from not having formed the conception of the action as a *function* of this kind, the consequences that have been here deduced from the formula (A.) for the variation of that definite integral, appear to have escaped the notice of **LAGRANGE**, and of the other illustrious analysts who have written on theoretical mechanics ; although they were in possession of a formula for the variation of this integral not greatly differing from ours. For although **LAGRANGE** and others, in treating of the motion of a system, have shown that the variation of this definite integral vanishes when the extreme coordinates and the constant H are given, they appear to have deduced from this result only the well known law of *least action* ; namely, that if the points or bodies of a system be imagined to move from a given set of initial to a given set of final positions, not as they do nor even as they could move consistently with the general dynamical laws or differential equations of motion, but so as not to violate any supposed geometrical connexions, nor that one dynamical relation between velocities and configurations which constitutes the law of living force ; and if, besides, this geometrically imaginable, but dynamically impossible motion, be made to differ infinitely *little* from the actual manner of motion of the system, between the given extreme positions ; then the varied value of the definite integral called action, or the accumulated living force of the system in the motion thus imagined, will differ infinitely *less* from the actual value of that integral. But when this well known law of least, or as it might be better called, of *stationary action*, is applied to the determination of the actual motion of a system, it serves only to form, by the rules of the calculus of variations, the differential equations of motion of the second order, which can always be otherwise found. It seems, therefore, to be with reason that **LAGRANGE**, **LAPLACE**, and **POISSON** have spoken lightly of the utility of this principle in the present state of dynamics. A different estimate, perhaps, will be formed of that

other principle which has been introduced in the present paper, under the name of the *law of varying action*, in which we pass from an actual motion to another motion dynamically possible, by varying the extreme positions of the system, and (in general) the quantity H_1 , and which serves to express, by means of a single function, not the mere differential equations of motion, but their intermediate and their final integrals.

Verifications of the foregoing Integrals.

4. A verification, which ought not to be neglected, and at the same time an illustration of this new principle, may be obtained by deducing the known differential equations of motion from our system of intermediate integrals, and by showing the consistence of these again with our final integral system. As preliminary to such verification, it is useful to observe that the final equation (6.) of living force, when combined with the system (C.), takes this new form,

$$\frac{1}{2} \sum \frac{1}{m} \left\{ \left(\frac{\partial V}{\partial x} \right)^2 + \left(\frac{\partial V}{\partial y} \right)^2 + \left(\frac{\partial V}{\partial z} \right)^2 \right\} = U + H; \quad \dots \quad (F.)$$

and that the initial equation (7.) of living force becomes by (D.)

$$\frac{1}{2} \sum \frac{1}{m} \left\{ \left(\frac{\partial V}{\partial a} \right)^2 + \left(\frac{\partial V}{\partial b} \right)^2 + \left(\frac{\partial V}{\partial c} \right)^2 \right\} = U_0 + H. \quad \dots \quad (G.)$$

These two partial differential equations, initial and final, of the first order and the second degree, must both be identically satisfied by the characteristic function V : they furnish (as we shall find) the principal means of discovering the form of that function, and are of essential importance in its theory. If the form of this function were known, we might eliminate $3n - 1$ of the $3n$ initial coordinates between the $3n$ equations (C.); and although we cannot yet perform the actual process of this elimination, we are entitled to assert that it would remove along with the others the remaining initial coordinate, and would conduct to the equation (6.) of final living force, which might then be transformed into the equation (F.). In like manner we may conclude that all the $3n$ final coordinates could be eliminated together from the $3n$ equations (D.), and that the result would be the initial equation (7.) of living force, or the transformed equation (G.). We may therefore consider the law of living force, which assisted us in discovering the properties of our characteristic function V , as included reciprocally in those properties, and as resulting by elimination, in every particular case, from the systems (C.) and (D.); and in treating of either of these systems, or in conducting any other dynamical investigation by the method of this characteristic function, we are at liberty to employ the partial differential equations (F.) and (G.), which that function must necessarily satisfy.

It will now be easy to deduce, as we proposed, the known equations of motion (3.) of the second order, by differentiation and elimination of constants, from our interme-

diate integral system (C.), (E.), or even from a part of that system, namely, from the group (C.), when combined with the equation (F.). For we thus obtain

$$\left. \begin{aligned}
 m_1 x''_1 &= \frac{d}{dt} \frac{\delta V}{\delta x_1} = x'_1 \frac{\delta^2 V}{\delta x_1^2} + x'_2 \frac{\delta^2 V}{\delta x_1 \delta x_2} + \dots + x'_n \frac{\delta^2 V}{\delta x_1 \delta x_n} \\
 &\quad + y'_1 \frac{\delta^2 V}{\delta x_1 \delta y_1} + y'_2 \frac{\delta^2 V}{\delta x_1 \delta y_2} + \dots + y'_n \frac{\delta^2 V}{\delta x_1 \delta y_n} \\
 &\quad + z'_1 \frac{\delta^2 V}{\delta x_1 \delta z_1} + z'_2 \frac{\delta^2 V}{\delta x_1 \delta z_2} + \dots + z'_n \frac{\delta^2 V}{\delta x_1 \delta z_n} \\
 &= \frac{1}{m_1} \frac{\delta V}{\delta x_1} \frac{\delta^2 V}{\delta x_1^2} + \frac{1}{m_2} \frac{\delta V}{\delta x_2} \frac{\delta^2 V}{\delta x_1 \delta x_2} + \dots + \frac{1}{m_n} \frac{\delta V}{\delta x_n} \frac{\delta^2 V}{\delta x_1 \delta x_n} \\
 &\quad + \frac{1}{m_1} \frac{\delta V}{\delta y_1} \frac{\delta^2 V}{\delta x_1 \delta y_1} + \frac{1}{m_2} \frac{\delta V}{\delta y_2} \frac{\delta^2 V}{\delta x_1 \delta y_2} + \dots + \frac{1}{m_n} \frac{\delta V}{\delta y_n} \frac{\delta^2 V}{\delta x_1 \delta y_n} \\
 &\quad + \frac{1}{m_1} \frac{\delta V}{\delta z_1} \frac{\delta^2 V}{\delta x_1 \delta z_1} + \frac{1}{m_2} \frac{\delta V}{\delta z_2} \frac{\delta^2 V}{\delta x_1 \delta z_2} + \dots + \frac{1}{m_n} \frac{\delta V}{\delta z_n} \frac{\delta^2 V}{\delta x_1 \delta z_n} \\
 &= \frac{\delta}{\delta x_1} \sum \cdot \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x} \right)^2 + \left(\frac{\delta V}{\delta y} \right)^2 + \left(\frac{\delta V}{\delta z} \right)^2 \right\} = \frac{\delta}{\delta x} (U + H);
 \end{aligned} \right\} \quad \dots \quad (11)$$

that is, we obtain

$$m_1 x''_1 = \frac{\delta U}{\delta x_1}; \quad \dots \quad (12.)$$

And in like manner we might deduce, by differentiation, from the integrals (C.) and from (F.) all the other known differential equations of motion, of the second order, contained in the set marked (3.) ; or, more concisely, we may deduce at once the formula (1.), which contains all those known equations, by observing that the intermediate integrals (C.), when combined with the relation (F.), give

$$\left. \begin{aligned}
 \Sigma \cdot m (x'' \delta x + y'' \delta y + z'' \delta z) &= \Sigma \left(\frac{d}{dt} \frac{\delta V}{\delta x} \cdot \delta x + \frac{d}{dt} \frac{\delta V}{\delta y} \cdot \delta y + \frac{d}{dt} \frac{\delta V}{\delta z} \cdot \delta z \right) \\
 &= \Sigma \cdot \frac{1}{m} \left(\frac{\delta V}{\delta x} \frac{\delta}{\delta x} + \frac{\delta V}{\delta y} \frac{\delta}{\delta y} + \frac{\delta V}{\delta z} \frac{\delta}{\delta z} \right) \Sigma \left(\frac{\delta V}{\delta x} \delta x + \frac{\delta V}{\delta y} \delta y + \frac{\delta V}{\delta z} \delta z \right) \\
 &= \Sigma \left(\delta x \frac{\delta}{\delta x} + \delta y \frac{\delta}{\delta y} + \delta z \frac{\delta}{\delta z} \right) \Sigma \cdot \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x} \right)^2 + \left(\frac{\delta V}{\delta y} \right)^2 + \left(\frac{\delta V}{\delta z} \right)^2 \right\} \\
 &= \Sigma \left(\delta x \frac{\delta}{\delta x} + \delta y \frac{\delta}{\delta y} + \delta z \frac{\delta}{\delta z} \right) (U + H) \\
 &= \delta U.
 \end{aligned} \right\} \quad \dots \quad (13.)$$

5. Again, we were to show that our intermediate integral system, composed of the equations (C.) and (E.), with the $3n$ arbitrary constants $a_1, b_1, c_1, \dots, a_n, b_n, c_n$, (and involving also the auxiliary constant H), is consistent with our final integral system of equations (D.) and (E.), which contain $3n$ other arbitrary constants, namely, $a'_1, b'_1, c'_1, \dots, a'_n, b'_n, c'_n$. The immediate differentials of the equations (C.), (D.), (E.), taken with respect to the time, are, for the first group,

$$\left. \begin{aligned} \frac{d}{dt} \frac{\delta V}{\delta x_1} &= m_1 x''_1; \quad \frac{d}{dt} \frac{\delta V}{\delta x_2} = m_2 x''_2; \quad \dots \frac{d}{dt} \frac{\delta V}{\delta x_n} = m_n x''_n; \\ \frac{d}{dt} \frac{\delta V}{\delta y_1} &= m_1 y''_1; \quad \frac{d}{dt} \frac{\delta V}{\delta y_2} = m_2 y''_2; \quad \dots \frac{d}{dt} \frac{\delta V}{\delta y_n} = m_n y''_n; \\ \frac{d}{dt} \frac{\delta V}{\delta z_1} &= m_1 z''_1; \quad \frac{d}{dt} \frac{\delta V}{\delta z_2} = m_2 z''_2; \quad \dots \frac{d}{dt} \frac{\delta V}{\delta z_n} = m_n z''_n; \end{aligned} \right\} \dots \dots \dots \quad (H.)$$

for the second group,

$$\left. \begin{aligned} \frac{d}{dt} \frac{\delta V}{\delta a_1} &= 0; \quad \frac{d}{dt} \frac{\delta V}{\delta a_2} = 0; \quad \dots \frac{d}{dt} \frac{\delta V}{\delta a_n} = 0; \\ \frac{d}{dt} \frac{\delta V}{\delta b_1} &= 0; \quad \frac{d}{dt} \frac{\delta V}{\delta b_2} = 0; \quad \dots \frac{d}{dt} \frac{\delta V}{\delta b_n} = 0; \\ \frac{d}{dt} \frac{\delta V}{\delta c_1} &= 0; \quad \frac{d}{dt} \frac{\delta V}{\delta c_2} = 0; \quad \dots \frac{d}{dt} \frac{\delta V}{\delta c_n} = 0; \end{aligned} \right\} \dots \dots \dots \dots \quad (I.)$$

and finally, for the last equation,

$$\frac{d}{dt} \frac{\delta V}{\delta H} = 1. \quad \dots \dots \dots \dots \quad (K.)$$

By combining the equations (C.) with their differentials (H.), and with the relation (F.), we deduced, in the foregoing number, the known equations of motion (3.); and we are now to show the consistence of the same intermediate integrals (C.) with the group of differentials (I.), which have been deduced from the final integrals.

The first equation of the group (I.) may be developed thus:

$$\left. \begin{aligned} 0 &= x'_1 \frac{\delta^2 V}{\delta a_1 \delta x_1} + x'_2 \frac{\delta^2 V}{\delta a_1 \delta x_2} + \dots + x'_n \frac{\delta^2 V}{\delta a_1 \delta x_n} \\ &+ y'_1 \frac{\delta^2 V}{\delta a_1 \delta y_1} + y'_2 \frac{\delta^2 V}{\delta a_1 \delta y_2} + \dots + y'_n \frac{\delta^2 V}{\delta a_1 \delta y_n} \\ &+ z'_1 \frac{\delta^2 V}{\delta a_1 \delta z_1} + z'_2 \frac{\delta^2 V}{\delta a_1 \delta z_2} + \dots + z'_n \frac{\delta^2 V}{\delta a_1 \delta z_n}; \end{aligned} \right\} \dots \dots \dots \quad (14.)$$

and the others may be similarly developed. In order, therefore, to show that they are satisfied by the group (C.), it is sufficient to prove that the following equations are true,

$$\left. \begin{aligned} 0 &= \frac{\delta}{\delta a_i} \sum \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x} \right)^2 + \left(\frac{\delta V}{\delta y} \right)^2 + \left(\frac{\delta V}{\delta z} \right)^2 \right\}, \\ 0 &= \frac{\delta}{\delta b_i} \sum \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x} \right)^2 + \left(\frac{\delta V}{\delta y} \right)^2 + \left(\frac{\delta V}{\delta z} \right)^2 \right\}, \\ 0 &= \frac{\delta}{\delta c_i} \sum \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x} \right)^2 + \left(\frac{\delta V}{\delta y} \right)^2 + \left(\frac{\delta V}{\delta z} \right)^2 \right\}, \end{aligned} \right\} \dots \dots \dots \quad (L.)$$

the integer i receiving any value from 1 to n inclusive; which may be shown at once, and the required verification thereby be obtained, if we merely take the variation of the relation (F.) with respect to the initial coordinates, as in the former verification

we took its variation with respect to the final coordinates, and so obtained results which agreed with the known equations of motion, and which may be thus collected,

$$\left. \begin{aligned} \frac{\delta}{\delta x_i} \Sigma \cdot \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x} \right)^2 + \left(\frac{\delta V}{\delta y} \right)^2 + \left(\frac{\delta V}{\delta z} \right)^2 \right\} &= \frac{\delta U}{\delta x_i}; \\ \frac{\delta}{\delta y_i} \Sigma \cdot \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x} \right)^2 + \left(\frac{\delta V}{\delta y} \right)^2 + \left(\frac{\delta V}{\delta z} \right)^2 \right\} &= \frac{\delta U}{\delta y_i}; \\ \frac{\delta}{\delta z_i} \Sigma \cdot \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x} \right)^2 + \left(\frac{\delta V}{\delta y} \right)^2 + \left(\frac{\delta V}{\delta z} \right)^2 \right\} &= \frac{\delta U}{\delta z_i}. \end{aligned} \right\} \quad \text{(M.)}$$

The same relation (F.), by being varied with respect to the quantity H , conducts to the expression

$$\frac{\delta}{\delta H} \Sigma \cdot \frac{1}{2m} \left\{ \left(\frac{\delta V}{\delta x} \right)^2 + \left(\frac{\delta V}{\delta y} \right)^2 + \left(\frac{\delta V}{\delta z} \right)^2 \right\} = 1; \quad \text{(N.)}$$

and this, when developed, agrees with the equation (K.), which is a new verification of the consistence of our foregoing results. Nor would it have been much more difficult, by the help of the foregoing principles, to have integrated directly our integrals of the first order, and so to have deduced in a different way our final integral system.

6. It may be considered as still another verification of our own general integral equations, to show that they include not only the known law of living force, or the integral expressing that law, but also the six other known integrals of the first order, which contain the law of motion of the centre of gravity, and the law of description of areas. For this purpose, it is only necessary to observe that it evidently follows from the conception of our characteristic function V , that this function depends on the initial and final positions of the attracting or repelling points of a system, not as referred to any foreign standard, but only as compared with one another; and therefore that this function will not vary, if without making any real change in either initial or final configuration, or in the relation of these to each other, we alter at once all the initial and all the final positions of the points of the system, by any common motion, whether of translation or of rotation. Now by considering three coordinate translations, we obtain the three following partial differential equations of the first order, which the function V must satisfy,

$$\left. \begin{aligned} \Sigma \frac{\delta V}{\delta x} + \Sigma \frac{\delta V}{\delta a} &= 0; \\ \Sigma \frac{\delta V}{\delta y} + \Sigma \frac{\delta V}{\delta b} &= 0; \\ \Sigma \frac{\delta V}{\delta z} + \Sigma \frac{\delta V}{\delta c} &= 0; \end{aligned} \right\} \quad \text{(O.)}$$

and by considering three coordinate rotations, we obtain these three other relations between the partial differential coefficients of the same order of the same characteristic function,

$$\left. \begin{array}{l} \Sigma \left(x \frac{\delta V}{\delta y} - y \frac{\delta V}{\delta x} \right) + \Sigma \left(a \frac{\delta V}{\delta b} - b \frac{\delta V}{\delta a} \right) = 0; \\ \Sigma \left(y \frac{\delta V}{\delta z} - z \frac{\delta V}{\delta y} \right) + \Sigma \left(b \frac{\delta V}{\delta c} - c \frac{\delta V}{\delta b} \right) = 0; \\ \Sigma \left(z \frac{\delta V}{\delta x} - x \frac{\delta V}{\delta z} \right) + \Sigma \left(c \frac{\delta V}{\delta a} - a \frac{\delta V}{\delta c} \right) = 0; \end{array} \right\} \quad \dots \quad (P.)$$

and if we change the final coefficients of V to the final components of momentum, and the initial coefficients to the initial components taken negatively, according to the dynamical properties of this function expressed by the integrals (C.) and (D.), we shall change these partial differential equations (O.) (P.), to the following,

$$\Sigma . m x' = \Sigma . m a'; \quad \Sigma . m y' = \Sigma . m b'; \quad \Sigma . m z' = \Sigma . m c'; \quad \dots \quad (15.)$$

and

$$\left. \begin{array}{l} \Sigma . m (x y' - y x') = \Sigma . m (a b' - b a'); \\ \Sigma . m (y z' - z y') = \Sigma . m (b c' - c b'); \\ \Sigma . m (z x' - x z') = \Sigma . m (c a' - a c'). \end{array} \right\} \quad \dots \quad (16.)$$

In this manner, therefore, we can deduce from the properties of our characteristic function the six other known integrals above mentioned, in addition to that seventh which contains the law of living force, and which assisted in the discovery of our method.

Introduction of relative or polar Coordinates, or other marks of position of a System.

7. The property of our characteristic function, by which it depends only on the internal or mutual relations between the positions initial and final of the points of an attracting or repelling system, suggests an advantage in employing internal or relative coordinates; and from the analogy of other applications of algebraical methods to researches of a geometrical kind, it may be expected that polar and other marks of position will also often be found useful. Supposing, therefore, that the $3n$ final coordinates $x_1 y_1 z_1 \dots x_n y_n z_n$ have been expressed as functions of $3n$ other variables, $\eta_1 \eta_2 \dots \eta_{3n}$, and that the $3n$ initial coordinates have in like manner been expressed as functions of $3n$ similar quantities, which we shall call $e_1 e_2 \dots e_{3n}$, we shall proceed to assign a general method for introducing these new marks of position into the expressions of our fundamental relations.

For this purpose we have only to transform the law of varying action, or the fundamental formula (A.), by transforming the two sums,

$$\Sigma . m (x' \delta x + y' \delta y + z' \delta z), \text{ and } \Sigma . m (a' \delta a + b' \delta b + c' \delta c),$$

which it involves, and which are respectively equivalent to the following more developed expressions,

$$\begin{aligned} \Sigma \cdot m (x' \delta x + y' \delta y + z' \delta z) = & m_1 (x'_1 \delta x_1 + y'_1 \delta y_1 + z'_1 \delta z_1) \\ & + m_2 (x'_2 \delta x_2 + y'_2 \delta y_2 + z'_2 \delta z_2) \\ & + \text{etc.} + m_n (x'_n \delta x_n + y'_n \delta y_n + z'_n \delta z_n); \end{aligned} \quad \quad (17.)$$

$$\begin{aligned} \Sigma \cdot m (a' \delta a + b' \delta b + c' \delta c) = & m_1 (a'_1 \delta a_1 + b'_1 \delta b_1 + c'_1 \delta c_1) \\ & + m_2 (a'_2 \delta a_2 + b'_2 \delta b_2 + c'_2 \delta c_2) \\ & + \text{etc.} + m_n (a'_n \delta a_n + b'_n \delta b_n + c'_n \delta c_n). \end{aligned} \quad \quad (18.)$$

Now x_i being by supposition a function of the $3n$ new marks of position $\eta_1 \dots \eta_{3n}$, its variation δx_i , and its differential coefficient x'_i , may be thus expressed:

$$\delta x_i = \frac{\delta x_i}{\delta \eta_1} \delta \eta_1 + \frac{\delta x_i}{\delta \eta_2} \delta \eta_2 + \dots + \frac{\delta x_i}{\delta \eta_{3n}} \delta \eta_{3n}; \quad \dots \quad (19.)$$

$$x'_i = \frac{\delta x_i}{\delta \eta'_1} \eta'_1 + \frac{\delta x_i}{\delta \eta'_2} \eta'_2 + \dots + \frac{\delta x_i}{\delta \eta'_{3n}} \eta'_{3n}; \quad \dots \quad (20.)$$

and similarly for y_i and z_i . If, then, we consider x'_i as a function, by (20.), of $\eta'_1 \dots \eta'_{3n}$, involving also in general $\eta_1 \dots \eta_{3n}$, and if we take its partial differential coefficients of the first order with respect to $\eta'_1 \dots \eta'_{3n}$, we find the relations,

$$\frac{\delta x'_i}{\delta \eta'_1} = \frac{\delta x_i}{\delta \eta_1}; \quad \frac{\delta x'_i}{\delta \eta'_2} = \frac{\delta x_i}{\delta \eta_2}; \quad \dots \quad \frac{\delta x'_i}{\delta \eta'_{3n}} = \frac{\delta x_i}{\delta \eta_{3n}}; \quad \dots \quad (21.)$$

and therefore we obtain these new expressions for the variations δx_i , δy_i , δz_i ,

$$\begin{aligned} \delta x_i = & \frac{\delta x'_i}{\delta \eta'_1} \delta \eta_1 + \frac{\delta x'_i}{\delta \eta'_2} \delta \eta_2 + \dots + \frac{\delta x'_i}{\delta \eta'_{3n}} \delta \eta_{3n}, \\ \delta y_i = & \frac{\delta y'_i}{\delta \eta'_1} \delta \eta_1 + \frac{\delta y'_i}{\delta \eta'_2} \delta \eta_2 + \dots + \frac{\delta y'_i}{\delta \eta'_{3n}} \delta \eta_{3n}, \\ \delta z_i = & \frac{\delta z'_i}{\delta \eta'_1} \delta \eta_1 + \frac{\delta z'_i}{\delta \eta'_2} \delta \eta_2 + \dots + \frac{\delta z'_i}{\delta \eta'_{3n}} \delta \eta_{3n}. \end{aligned} \quad \quad (22.)$$

Substituting these expressions (22.) for the variations in the sum (17.), we easily transform it into the following,

$$\begin{aligned} \Sigma \cdot m (x' \delta x + y' \delta y + z' \delta z) = & \Sigma \cdot m \left(x' \frac{\delta x'}{\delta \eta'_1} + y' \frac{\delta y'}{\delta \eta'_1} + z' \frac{\delta z'}{\delta \eta'_1} \right) \cdot \delta \eta_1 \\ & + \Sigma \cdot m \left(x' \frac{\delta x'}{\delta \eta'_2} + y' \frac{\delta y'}{\delta \eta'_2} + z' \frac{\delta z'}{\delta \eta'_2} \right) \cdot \delta \eta_2 \\ & + \text{etc.} + \Sigma \cdot m \left(x' \frac{\delta x'}{\delta \eta'_{3n}} + y' \frac{\delta y'}{\delta \eta'_{3n}} + z' \frac{\delta z'}{\delta \eta'_{3n}} \right) \cdot \delta \eta_{3n} \\ & = \frac{\delta T}{\delta \eta'_1} \delta \eta_1 + \frac{\delta T}{\delta \eta'_2} \delta \eta_2 + \dots + \frac{\delta T}{\delta \eta'_{3n}} \delta \eta_{3n}; \end{aligned} \quad \quad (23.)$$

T being the same quantity as before, namely, the half of the final living force of the

system, but being now considered as a function of $\eta'_1 \dots \eta'_{3n}$, involving also the masses, and in general $\eta_1 \dots \eta_{3n}$, and obtained by substituting for the quantities $x' y' z'$ their values of the form (20.) in the equation of definition

$$T = \frac{1}{2} \sum m (x'^2 + y'^2 + z'^2). \quad \dots \dots \dots \quad (4.)$$

In like manner we find this transformation for the sum (18.),

$$\sum m (a' \delta a + b' \delta b + c' \delta c) = \frac{\delta T_0}{\delta e'_1} \delta e_1 + \frac{\delta T_0}{\delta e'_2} \delta e_2 + \dots + \frac{\delta T_0}{\delta e'_{3n}} \delta e_{3n}. \quad \dots \quad (24.)$$

The law of varying action, or the formula (A.), becomes therefore, when expressed by the present more general coordinates or marks of position,

$$\delta V = \sum \frac{\delta T}{\delta \eta'} \delta \eta - \sum \frac{\delta T_0}{\delta e'} \delta e + t \delta H; \quad \dots \dots \dots \quad (Q.)$$

and instead of the groups (C.) and (D.), into which, along with the equation (E.), this law resolved itself before, it gives now these other groups,

$$\frac{\delta V}{\delta \eta_1} = \frac{\delta T}{\delta \eta'_1}; \quad \frac{\delta V}{\delta \eta_2} = \frac{\delta T}{\delta \eta'_2}; \quad \dots \quad \frac{\delta V}{\delta \eta_{3n}} = \frac{\delta T}{\delta \eta'_{3n}}; \quad \dots \dots \dots \quad (R.)$$

and

$$\frac{\delta V}{\delta e_1} = - \frac{\delta T_0}{\delta e'_1}; \quad \frac{\delta V}{\delta e_2} = - \frac{\delta T_0}{\delta e'_2}; \quad \dots \quad \frac{\delta V}{\delta e_{3n}} = - \frac{\delta T_0}{\delta e'_{3n}}. \quad \dots \quad (S.)$$

The quantities $e_1 e_2 \dots e_{3n}$ and $e'_1 e'_2 \dots e'_{3n}$ are now the initial data respecting the manner of motion of the system; and the $3n$ final integrals, connecting these $6n$ initial data, and the n masses, with the time t , and with the $3n$ final or varying quantities $\eta_1 \eta_2 \dots \eta_{3n}$, which mark the varying positions of the n moving points of the system, are now to be obtained by eliminating the auxiliary constant H between the $3n+1$ equations (S.) and (E.); while the $3n$ intermediate integrals, or integrals of the first order, which connect the same varying marks of position and their first differential coefficients with the time, the masses, and the initial marks of position, are the result of elimination of the same auxiliary constant H between the equations (R.) and (E.). Our fundamental formula, and intermediate and final integrals, can therefore be very simply expressed with any new sets of coordinates; and the partial differential equations (F.) (G.), which our characteristic function V must satisfy, and which are, as we have said, essential in the theory of that function, can also easily be expressed with any such transformed coordinates, by merely combining the final and initial expressions of the law of living force,

$$T = U + H, \quad \dots \dots \dots \quad (6.)$$

$$T_0 = U_0 + H, \quad \dots \dots \dots \quad (7.)$$

with the new groups (R.) and (S.). For this purpose we must now consider the function U , of the masses and mutual distances of the several points of the system, as depending on the new marks of position $\eta_1 \eta_2 \dots \eta_{3n}$; and the analogous function U_0 , as depending similarly on the initial quantities $e_1 e_2 \dots e_{3n}$; we must also suppose

that T is expressed (as it may) as a function of its own coefficients $\frac{\delta T}{\delta \eta'_1}, \frac{\delta T}{\delta \eta'_2}, \dots, \frac{\delta T}{\delta \eta'_{3n}}$,

which will always be, with respect to these, homogeneous of the second dimension, and may also involve explicitly the quantities $\eta_1 \eta_2 \dots \eta_{3n}$; and that T_0 is expressed

as a similar function of its coefficients $\frac{\delta T_0}{\delta e'_1}, \frac{\delta T_0}{\delta e'_2}, \dots, \frac{\delta T_0}{\delta e'_{3n}}$; so that

$$\left. \begin{aligned} T &= F \left(\frac{\delta T}{\delta \eta'_1}, \frac{\delta T}{\delta \eta'_2}, \dots, \frac{\delta T}{\delta \eta'_{3n}} \right), \\ T_0 &= F \left(\frac{\delta T_0}{\delta e'_1}, \frac{\delta T_0}{\delta e'_2}, \dots, \frac{\delta T_0}{\delta e'_{3n}} \right); \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (25.)$$

and that then these coefficients of T and T_0 are changed to their values (R.) and (S.), so as to give, instead of (F.) and (G.), two other transformed equations, namely,

$$F \left(\frac{\delta V}{\delta \eta'_1}, \frac{\delta V}{\delta \eta'_2}, \dots, \frac{\delta V}{\delta \eta'_{3n}} \right) = U + H, \quad \dots \quad \dots \quad \dots \quad (T.)$$

and, on account of the homogeneity and dimension of T_0 ,

$$F \left(\frac{\delta V}{\delta e'_1}, \frac{\delta V}{\delta e'_2}, \dots, \frac{\delta V}{\delta e'_{3n}} \right) = U_0 + H. \quad \dots \quad \dots \quad \dots \quad (U.)$$

8. Nor is there any difficulty in deducing analogous transformations for the known differential equations of motion of the second order, of any system of free points, by taking the variation of the new form (T.) of the law of living force, and by attending to the dynamical meanings of the coefficients of our characteristic function. For if we observe that the final living force $2T$, when considered as a function of $\eta_1 \eta_2 \dots \eta_{3n}$ and of $\eta'_1 \eta'_2 \dots \eta'_{3n}$, is necessarily homogeneous of the second dimension with respect to the latter set of variables, and must therefore satisfy the condition

$$2T = \eta'_1 \frac{\delta T}{\delta \eta'_1} + \eta'_2 \frac{\delta T}{\delta \eta'_2} + \dots + \eta'_{3n} \frac{\delta T}{\delta \eta'_{3n}}, \quad \dots \quad \dots \quad \dots \quad (26.)$$

we shall perceive that its total variation,

$$\left. \begin{aligned} \delta T &= \frac{\delta T}{\delta \eta'_1} \delta \eta_1 + \frac{\delta T}{\delta \eta'_2} \delta \eta_2 + \dots + \frac{\delta T}{\delta \eta'_{3n}} \delta \eta_{3n} \\ &\quad + \frac{\delta T}{\delta \eta'_1} \delta \eta'_1 + \frac{\delta T}{\delta \eta'_2} \delta \eta'_2 + \dots + \frac{\delta T}{\delta \eta'_{3n}} \delta \eta'_{3n}, \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad (27.)$$

may be put under the form

$$\left. \begin{aligned} \delta T &= \eta'_1 \delta \frac{\delta T}{\delta \eta'_1} + \eta'_2 \delta \frac{\delta T}{\delta \eta'_2} + \dots + \eta'_{3n} \delta \frac{\delta T}{\delta \eta'_{3n}} \\ &\quad - \frac{\delta T}{\delta \eta'_1} \delta \eta_1 - \frac{\delta T}{\delta \eta'_2} \delta \eta_2 - \dots - \frac{\delta T}{\delta \eta'_{3n}} \delta \eta_{3n} \\ &= \Sigma \eta' \delta \frac{\delta T}{\delta \eta'} - \Sigma \frac{\delta T}{\delta \eta} \delta \eta \\ &= \Sigma \left(\eta' \delta \frac{\delta V}{\delta \eta} - \frac{\delta T}{\delta \eta} \delta \eta \right), \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad (28.)$$

and therefore that the total variation of the new partial differential equation (T.) may be thus written,

$$\Sigma \left(\eta' \delta \frac{\delta V}{\delta \eta} - \frac{\delta T}{\delta \eta} \delta \eta \right) = \Sigma \cdot \frac{\delta U}{\delta \eta} \delta \eta + \delta H: \quad \dots \quad (V.)$$

in which, if we observe that $\eta' = \frac{d \eta}{d t}$, and that the quantities of the form η are the only ones which vary with the time, we shall see that

$$\Sigma \cdot \eta' \delta \frac{\delta V}{\delta \eta} = \Sigma \left(\frac{d}{dt} \frac{\delta V}{\delta \eta} \cdot \delta \eta + \frac{d}{dt} \frac{\delta V}{\delta e} \cdot \delta e \right) + \frac{d}{dt} \frac{\delta V}{\delta H} \cdot \delta H, \quad \dots \quad (29.)$$

because the identical equation $\delta dV = d\delta V$ gives, when developed,

$$\left. \begin{aligned} & \Sigma \left(\delta \frac{\delta V}{\delta \eta} \cdot d \eta + \delta \frac{\delta V}{\delta e} \cdot d e \right) + \delta \frac{\delta V}{\delta H} \cdot d H \\ & = \Sigma \left(d \frac{\delta V}{\delta \eta} \cdot \delta \eta + d \frac{\delta V}{\delta e} \cdot \delta e \right) + d \frac{\delta V}{\delta H} \cdot \delta H. \end{aligned} \right\} \quad \dots \quad (30.)$$

Decomposing, therefore, the expression (V.), for the variation of half the living force, into as many separate equations as it contains independent variations, we obtain, not only the equation

$$\frac{d}{dt} \frac{\delta V}{\delta H} = 1, \quad \dots \quad (K.)$$

which had already presented itself, and the group

$$\frac{d}{dt} \frac{\delta V}{\delta e_1} = 0, \quad \frac{d}{dt} \frac{\delta V}{\delta e_2} = 0, \dots \quad \frac{d}{dt} \frac{\delta V}{\delta e_{3n}} = 0, \quad \dots \quad (W.)$$

which might have been at once obtained by differentiation from the final integrals (S.), but also a group of $3n$ other equations of the form

$$\frac{d}{dt} \frac{\delta V}{\delta \eta} - \frac{\delta T}{\delta \eta} = \frac{\delta U}{\delta \eta}, \quad \dots \quad (X.)$$

which give, by the intermediate integrals (R.),

$$\frac{d}{dt} \frac{\delta T}{\delta \eta'} - \frac{\delta T}{\delta \eta} = \frac{\delta U}{\delta \eta}; \quad \dots \quad (Y.)$$

that is, more fully,

$$\left. \begin{aligned} & \frac{d}{dt} \frac{\delta T}{\delta \eta'_1} - \frac{\delta T}{\delta \eta_1} = \frac{\delta U}{\delta \eta_1}; \\ & \frac{d}{dt} \frac{\delta T}{\delta \eta'_2} - \frac{\delta T}{\delta \eta_2} = \frac{\delta U}{\delta \eta_2}; \\ & \dots \\ & \frac{d}{dt} \frac{\delta T}{\delta \eta'_{3n}} - \frac{\delta T}{\delta \eta_{3n}} = \frac{\delta U}{\delta \eta_{3n}}. \end{aligned} \right\} \quad \dots \quad (Z.)$$

These last transformations of the differential equations of motion of the second order, of an attracting or repelling system, coincide in all respects (a slight difference of notation excepted,) with the elegant canonical forms in the *Mécanique Analytique* of LAGRANGE; but it seemed worth while to deduce them here anew,

from the properties of our characteristic function. And if we were to suppose (as it has often been thought convenient and even necessary to do,) that the n points of a system are not entirely free, nor subject only to their own mutual attractions or repulsions, but connected by any geometrical conditions, and influenced by any foreign agencies, consistent with the law of conservation of living force; so that the number of independent marks of position should be now less numerous, and the force-function U less simple than before; it might still be proved, by a reasoning very similar to the foregoing, that on these suppositions also (which, however, the dynamical spirit is tending more and more to exclude,) the accumulated living force or action V of the system is a *characteristic motion-function* of the kind already explained; having the same law and formula of variation, which are susceptible of the same transformations; obliged to satisfy in the same way a final and an initial relation between its partial differential coefficients of the first order; conducting, by the variation of one of these two relations, to the same canonical forms assigned by LAGRANGE for the differential equations of motion; and furnishing, on the same principles as before, their intermediate and their final integrals. To those imaginable cases, indeed, in which the law of living force no longer holds, our method also would not apply; but it appears to be the growing conviction of the persons who have meditated the most profoundly on the mathematical dynamics of the universe, that these are cases suggested by insufficient views of the mutual actions of body.

9. It results from the foregoing remarks, that in order to apply our method of the characteristic function to any problem of dynamics respecting any moving system, the known law of living force is to be combined with our law of varying action; and that the general expression of this latter law is to be obtained in the following manner. We are first to express the quantity T , namely, the half of the living force of the system, as a function (which will always be homogeneous of the second dimension,) of the differential coefficients or rates of increase $\eta'_1, \eta'_2, \&c.$, of any rectangular coordinates, or other marks of position of the system: we are next to take the variation of this homogeneous function with respect to those rates of increase, and to change the variations of those rates $\delta \eta'_1, \delta \eta'_2, \&c.$, to the variations $\delta \eta_1, \delta \eta_2, \&c.$, of the marks of position themselves; and then to subtract the initial from the final value of the result, and to equate the remainder to $\delta V - t \delta H$. A slight consideration will show that this general rule or process for obtaining the variation of the characteristic function V , is applicable even when the marks of position $\eta_1, \eta_2, \&c.$, are not all independent of each other; which will happen when they have been made, from any motive of convenience, more numerous than the rectangular coordinates of the several points of the system. For if we suppose that the $3n$ rectangular coordinates $x_1 y_1 z_1 \dots x_n y_n z_n$, have been expressed by any transformation as functions of $3n+k$ other marks of position, $\eta_1 \eta_2 \dots \eta_{3n+k}$, which must therefore be connected by k equations of condition,

$$\left. \begin{array}{l} 0 = \varphi_1(\eta_1, \eta_2, \dots, \eta_{3n+k}), \\ 0 = \varphi_2(\eta_1, \eta_2, \dots, \eta_{3n+k}), \\ \dots \\ 0 = \varphi_k(\eta_1, \eta_2, \dots, \eta_{3n+k}), \end{array} \right\} \dots \dots \dots \dots \dots \dots \quad (31.)$$

giving k of the new marks of position as functions of the remaining $3n$,

$$\left. \begin{array}{l} \eta_{3n+1} = \psi_1(\eta_1, \eta_2, \dots, \eta_{3n}), \\ \eta_{3n+2} = \psi_2(\eta_1, \eta_2, \dots, \eta_{3n}), \\ \dots \\ \eta_{3n+k} = \psi_k(\eta_1, \eta_2, \dots, \eta_{3n}), \end{array} \right\} \dots \dots \dots \dots \dots \dots \quad (32.)$$

the expression

$$T = \frac{1}{2} \sum m (x'^2 + y'^2 + z'^2), \dots \dots \dots \dots \dots \quad (4.)$$

will become, by the introduction of these new variables, a homogeneous function of the second dimension of the $3n+k$ rates of increase $\eta'_1, \eta'_2, \dots, \eta'_{3n+k}$, involving also in general $\eta_1, \eta_2, \dots, \eta_{3n+k}$ and having a variation which may be thus expressed:

$$\left. \begin{array}{l} \delta T = \left(\frac{\delta T}{\delta \eta'_1} \right) \delta \eta'_1 + \left(\frac{\delta T}{\delta \eta'_2} \right) \delta \eta'_2 + \dots + \left(\frac{\delta T}{\delta \eta'_{3n+k}} \right) \delta \eta'_{3n+k} \\ + \left(\frac{\delta T}{\delta \eta_1} \right) \delta \eta_1 + \left(\frac{\delta T}{\delta \eta_2} \right) \delta \eta_2 + \dots + \left(\frac{\delta T}{\delta \eta_{3n+k}} \right) \delta \eta_{3n+k}; \end{array} \right\} \dots \dots \quad (33.)$$

or in this other way,

$$\left. \begin{array}{l} \delta T = \frac{\delta T}{\delta \eta'_1} \delta \eta'_1 + \frac{\delta T}{\delta \eta'_2} \delta \eta'_2 + \dots + \frac{\delta T}{\delta \eta'_{3n}} \delta \eta'_{3n} \\ + \frac{\delta T}{\delta \eta_1} \delta \eta_1 + \frac{\delta T}{\delta \eta_2} \delta \eta_2 + \dots + \frac{\delta T}{\delta \eta_{3n}} \delta \eta_{3n}, \end{array} \right\} \dots \dots \dots \dots \quad (34.)$$

on account of the relations (32.), which give, when differentiated with respect to the time,

$$\left. \begin{array}{l} \eta'_{3n+1} = \eta'_1 \frac{\delta \psi_1}{\delta \eta_1} + \eta'_2 \frac{\delta \psi_1}{\delta \eta_2} + \dots + \eta'_{3n} \frac{\delta \psi_1}{\delta \eta_{3n}}, \\ \eta'_{3n+2} = \eta'_1 \frac{\delta \psi_2}{\delta \eta_1} + \eta'_2 \frac{\delta \psi_2}{\delta \eta_2} + \dots + \eta'_{3n} \frac{\delta \psi_2}{\delta \eta_{3n}}, \\ \dots \\ \eta'_{3n+k} = \eta'_1 \frac{\delta \psi_k}{\delta \eta_1} + \eta'_2 \frac{\delta \psi_k}{\delta \eta_2} + \dots + \eta'_{3n} \frac{\delta \psi_k}{\delta \eta_{3n}}, \end{array} \right\} \dots \dots \dots \dots \quad (35.)$$

and therefore, attending only to the variations of quantities of the form η' ,

$$\left. \begin{array}{l} \delta \eta'_{3n+1} = \frac{\delta \psi_1}{\delta \eta_1} \delta \eta'_1 + \frac{\delta \psi_1}{\delta \eta_2} \delta \eta'_2 + \dots + \frac{\delta \psi_1}{\delta \eta_{3n}} \delta \eta'_{3n}, \\ \delta \eta'_{3n+2} = \frac{\delta \psi_2}{\delta \eta_1} \delta \eta'_1 + \frac{\delta \psi_2}{\delta \eta_2} \delta \eta'_2 + \dots + \frac{\delta \psi_2}{\delta \eta_{3n}} \delta \eta'_{3n}, \\ \dots \\ \delta \eta'_{3n+k} = \frac{\delta \psi_k}{\delta \eta_1} \delta \eta'_1 + \frac{\delta \psi_k}{\delta \eta_2} \delta \eta'_2 + \dots + \frac{\delta \psi_k}{\delta \eta_{3n}} \delta \eta'_{3n}, \end{array} \right\} \dots \dots \dots \dots \quad (36.)$$

Comparing the two expressions (33.) and (34.), we find by (36.) the relations

$$\left. \begin{aligned}
 \frac{\delta T}{\delta \eta'_1} &= \left(\frac{\delta T}{\delta \eta'_1} \right) + \left(\frac{\delta T}{\delta \eta'_{3n+1}} \right) \frac{\delta \Psi_1}{\delta \eta_1} + \left(\frac{\delta T}{\delta \eta'_{3n+2}} \right) \frac{\delta \Psi_2}{\delta \eta_1} + \dots + \left(\frac{\delta T}{\delta \eta'_{3n+k}} \right) \frac{\delta \Psi_k}{\delta \eta_1}, \\
 \frac{\delta T}{\delta \eta'_2} &= \left(\frac{\delta T}{\delta \eta'_2} \right) + \left(\frac{\delta T}{\delta \eta'_{3n+1}} \right) \frac{\delta \Psi_1}{\delta \eta_2} + \left(\frac{\delta T}{\delta \eta'_{3n+2}} \right) \frac{\delta \Psi_2}{\delta \eta_2} + \dots + \left(\frac{\delta T}{\delta \eta'_{3n+k}} \right) \frac{\delta \Psi_k}{\delta \eta_2}, \\
 &\dots \\
 \frac{\delta T}{\delta \eta'_{3n}} &= \left(\frac{\delta T}{\delta \eta'_{3n}} \right) + \left(\frac{\delta T}{\delta \eta'_{3n+1}} \right) \frac{\delta \Psi_1}{\delta \eta_{3n}} + \left(\frac{\delta T}{\delta \eta'_{3n+2}} \right) \frac{\delta \Psi_2}{\delta \eta_{3n}} + \dots + \left(\frac{\delta T}{\delta \eta'_{3n+k}} \right) \frac{\delta \Psi_k}{\delta \eta_{3n}}; \end{aligned} \right\} (37.)$$

which give, by (32.),

$$\left. \begin{aligned} \frac{\delta T}{\delta \eta'_1} \delta \eta_1 + \frac{\delta T}{\delta \eta'_2} \delta \eta_2 + \dots + \frac{\delta T}{\delta \eta'_{3n}} \delta \eta_{3n} = \\ \left(\frac{\delta T}{\delta \eta'_1} \right) \delta \eta_1 + \left(\frac{\delta T}{\delta \eta'_2} \right) \delta \eta_2 + \dots + \left(\frac{\delta T}{\delta \eta'_{3n+k}} \right) \delta \eta_{3n+k}; \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (38.)$$

we may therefore put the expression (Q.) under the following more general form,

$$\delta V = \Sigma \cdot \left(\frac{\delta T}{\delta \eta'} \right) \delta \eta - \Sigma \cdot \left(\frac{\delta T_0}{\delta e'} \right) \delta e + t \delta H, \dots \quad (A^1)$$

the coefficients $\left(\frac{\delta T}{\delta \eta^j}\right)$ being formed by treating all the $3n + k$ quantities η'_1, η'_2, \dots

η'_{3n+k} , as independent; which was the extension above announced, of the rule for forming the variation of the characteristic function V .

We cannot, however, immediately decompose this new expression (A') for δV , as we did the expression (Q), by treating all the variations δz , δe , as independent; but we may decompose it so, if we previously combine it with the final equations of condition (31.), and with the analogous initial equations of condition, namely,

$$\left. \begin{array}{l} 0 = \Phi_1(e_1, e_2, \dots, e_{3n+k}), \\ 0 = \Phi_2(e_1, e_2, \dots, e_{3n+k}), \\ \dots \\ 0 = \Phi_k(e_1, e_2, \dots, e_{3n+k}), \end{array} \right\} \dots \dots \dots \dots \dots \dots \quad (39)$$

which we may do by adding the variations of the connecting functions $\varphi_1, \dots, \varphi_k$, Φ_1, \dots, Φ_k , multiplied respectively by factors to be determined, $\lambda_1, \dots, \lambda_k$, $\Delta_1, \dots, \Delta_k$. In this manner the law of varying action takes this new form,

$$\delta V = \Sigma \cdot \left(\frac{\delta T}{\delta \eta'} \right) \delta \eta - \Sigma \cdot \left(\frac{\delta T_0}{\delta e'} \right) \delta e + t \delta H + \Sigma \cdot \lambda \delta \varphi + \Sigma \cdot \Lambda \delta \Phi; \dots \quad (B^1)$$

and decomposes itself into $6n + 2k + 1$ separate expressions, for the partial differential coefficients of the first order of the characteristic function V , namely, into the following,

$$\left. \begin{aligned} \frac{\delta V}{\delta \eta_1} &= \left(\frac{\delta T}{\delta \eta'_1} \right) + \lambda_1 \frac{\delta \varphi_1}{\delta \eta_1} + \lambda_2 \frac{\delta \varphi_2}{\delta \eta_1} + \dots + \lambda_k \frac{\delta \varphi_k}{\delta \eta_1}, \\ \frac{\delta V}{\delta \eta_2} &= \left(\frac{\delta T}{\delta \eta'_2} \right) + \lambda_1 \frac{\delta \varphi_1}{\delta \eta_2} + \lambda_2 \frac{\delta \varphi_2}{\delta \eta_2} + \dots + \lambda_k \frac{\delta \varphi_k}{\delta \eta_2}, \\ &\dots \\ \frac{\delta V}{\delta \eta_{3n+k}} &= \left(\frac{\delta T}{\delta \eta'_{3n+k}} \right) + \lambda_1 \frac{\delta \varphi_1}{\delta \eta_{3n+k}} + \dots + \lambda_k \frac{\delta \varphi_k}{\delta \eta_{3n+k}}, \end{aligned} \right\} \dots \dots \dots \quad (C^1)$$

and

$$\left. \begin{aligned} \frac{\delta V}{\delta e_1} &= - \left(\frac{\delta T_0}{\delta e'_1} \right) + \Lambda_1 \frac{\delta \Phi_1}{\delta e_1} + \Lambda_2 \frac{\delta \Phi_2}{\delta e_1} + \dots + \Lambda_k \frac{\delta \Phi_k}{\delta e_1}, \\ \frac{\delta V}{\delta e_2} &= - \left(\frac{\delta T_0}{\delta e'_2} \right) + \Lambda_1 \frac{\delta \Phi_1}{\delta e_2} + \Lambda_2 \frac{\delta \Phi_2}{\delta e_2} + \dots + \Lambda_k \frac{\delta \Phi_k}{\delta e_2}, \\ &\dots \\ \frac{\delta V}{\delta e_{3n+k}} &= - \left(\frac{\delta T_0}{\delta e'_{3n+k}} \right) + \Lambda_1 \frac{\delta \Phi_1}{\delta e_{3n+k}} + \dots + \Lambda_k \frac{\delta \Phi_k}{\delta e_{3n+k}}, \end{aligned} \right\} \dots \dots \dots \quad (D^1)$$

besides the old equation (E.). The analogous introduction of multipliers in the canonical forms of LAGRANGE, for the differential equations of motion of the second order, by which a sum such as $\Sigma \lambda \frac{\delta \varphi}{\delta \eta}$ is added to $\frac{\delta U}{\delta \eta}$ in the second member of the formula (Y.), is also easily justified on the principles of the present essay.

Separation of the relative motion of a system from the motion of its centre of gravity ; characteristic function for such relative motion, and law of its variation.

10. As an example of the foregoing transformations, and at the same time as an important application, we shall now introduce relative coordinates, $x_i y_i z_i$, referred to an internal origin $x_u y_u z_u$; that is, we shall put

$$x_i = x_{i\mu} + x_{i\nu} \quad y_i = y_{i\mu} + y_{i\nu} \quad z_i = z_{i\mu} + z_{i\nu} \quad \dots \dots \dots \quad (40.)$$

and in like manner

$$a_i = a_{i\mu} + a_{i\nu} \quad b_i = b_{i\mu} + b_{i\nu} \quad c_i = c_{i\mu} + c_{i\nu}; \quad \dots \dots \dots \quad (41.)$$

together with the differentiated expressions

$$x'_i = x'_{i\mu} + x'_{i\nu} \quad y'_i = y'_{i\mu} + y'_{i\nu} \quad z'_i = z'_{i\mu} + z'_{i\nu} \quad \dots \dots \dots \quad (42.)$$

and

$$a'_i = a'_{i\mu} + a'_{i\nu} \quad b'_i = b'_{i\mu} + b'_{i\nu} \quad c'_i = c'_{i\mu} + c'_{i\nu} \quad \dots \dots \dots \quad (43.)$$

Introducing the expressions (42.) for the rectangular components of velocity, we find that the value given by (4.) for the living force $2T$, decomposes itself into the three following parts,

$$\begin{aligned} 2T &= \Sigma \cdot m (x'^\mu + y'^\mu + z'^\mu) = \Sigma \cdot m (x'^\mu + y'^\mu + z'^\mu) \\ &+ 2 (x'_u \Sigma \cdot m x'_i + y'_u \Sigma \cdot m y'_i + z'_u \Sigma \cdot m z'_i) + (x'^\mu + y'^\mu + z'^\mu) \Sigma m; \quad (44.) \end{aligned}$$

if then we establish, as we may, the three equations of condition,

$$\Sigma.m x_i = 0, \quad \Sigma.m y_i = 0, \quad \Sigma.m z_i = 0, \quad \dots \dots \dots \dots \quad (45.)$$

which give by (40.),

$$x_u = \frac{\Sigma.m x}{\Sigma m}, \quad y_u = \frac{\Sigma.m y}{\Sigma m}, \quad z_u = \frac{\Sigma.m z}{\Sigma m}, \quad \dots \dots \dots \dots \quad (46.)$$

so that $x_u y_u z_u$ are now the coordinates of the point which is called the centre of gravity of the system, we may reduce the function T to the form

$$T = T_i + T_u, \quad \dots \dots \dots \dots \dots \dots \dots \quad (47.)$$

in which

$$T_i = \frac{1}{2} \Sigma.m (x_i^2 + y_i^2 + z_i^2), \quad \dots \dots \dots \dots \dots \dots \dots \quad (48.)$$

and

$$T_u = \frac{1}{2} (x_u^2 + y_u^2 + z_u^2) \Sigma m. \quad \dots \dots \dots \dots \dots \dots \dots \quad (49.)$$

By this known decomposition, the whole living force $2T$ of the system is resolved into the two parts $2T_i$ and $2T_u$, of which the former, $2T_i$, may be called the *relative living force*, being that which results solely from the relative velocities of the points of the system, in their motions about their common centre of gravity $x_u y_u z_u$; while the latter part, $2T_u$, results only from the absolute motion of that centre of gravity in space, and is the same as if all the masses of the system were united in that common centre. At the same time, the law of living force, $T = U + H$, (6.), resolves itself by the law of motion of the centre of gravity into the two following separate equations,

$$T_i = U + H_i, \quad \dots \dots \dots \dots \dots \dots \dots \quad (50.)$$

and

$$T_u = H_u; \quad \dots \dots \dots \dots \dots \dots \dots \quad (51.)$$

H_i and H_u being two new constants independent of the time t , and such that their sum

$$H_i + H_u = H. \quad \dots \dots \dots \dots \dots \dots \dots \quad (52.)$$

And we may in like manner decompose the action, or accumulated living force V , which is equal to the definite integral $\int_0^t 2T dt$, into the two following analogous parts,

$$V = V_i + V_u, \quad \dots \dots \dots \dots \dots \dots \dots \quad (E^i.)$$

determined by the two equations,

$$V_i = \int_0^t 2T_i dt, \quad \dots \dots \dots \dots \dots \dots \dots \quad (F^i.)$$

and

$$V_u = \int_0^t 2T_u dt. \quad \dots \dots \dots \dots \dots \dots \dots \quad (G^i.)$$

The last equation gives by (51.),

$$V_u = 2H_u t; \quad \dots \dots \dots \dots \dots \dots \dots \quad (53.)$$

a result which, by the law of motion of the centre of gravity, may be thus expressed,

$$V_u = \sqrt{(x_u - a_u)^2 + (y_u - b_u)^2 + (z_u - c_u)^2} \cdot \sqrt{2H_u \Sigma m}; \quad \dots \dots \dots \dots \quad (H^i.)$$

a_u b_u c_u being the initial coordinates of the centre of gravity, so that

$$a_u = \frac{\Sigma \cdot m \cdot a}{\Sigma m}, \quad b_u = \frac{\Sigma \cdot m \cdot b}{\Sigma m}, \quad c_u = \frac{\Sigma \cdot m \cdot c}{\Sigma m} \quad \dots \dots \dots \dots \quad (54.)$$

And for the variation δV of the whole function V , the rule of the last number gives

$$\left. \begin{aligned} \delta V = & \Sigma \cdot m (x'_u \delta x_u - a'_u \delta a_u + y'_u \delta y_u - b'_u \delta b_u + z'_u \delta z_u - c'_u \delta c_u) \\ & + (x'_u \delta x_u - a'_u \delta a_u + y'_u \delta y_u - b'_u \delta b_u + z'_u \delta z_u - c'_u \delta c_u) \Sigma m \\ & + t \delta H + \lambda_1 \Sigma \cdot m \delta x_i + \lambda_2 \Sigma \cdot m \delta y_i + \lambda_3 \Sigma \cdot m \delta z_i \\ & + \Lambda_1 \Sigma \cdot m \delta a_i + \Lambda_2 \Sigma \cdot m \delta b_i + \Lambda_3 \Sigma \cdot m \delta c_i; \end{aligned} \right\} \quad (H^1.)$$

while the variation of the part V_u , determined by the equation $(H^1.)$, is easily shown to be equivalent to the part

$$\delta V_u = (x'_u \delta x_u - a'_u \delta a_u + y'_u \delta y_u - b'_u \delta b_u + z'_u \delta z_u - c'_u \delta c_u) \Sigma m + t \delta H_u; \quad (K^1.)$$

the variation of the other part V_i may therefore be thus expressed,

$$\left. \begin{aligned} \delta V_i = & \Sigma \cdot m (x'_i \delta x_i - a'_i \delta a_i + y'_i \delta y_i - b'_i \delta b_i + z'_i \delta z_i - c'_i \delta c_i) \\ & + t \delta H_i + \lambda_1 \Sigma \cdot m \delta x_i + \lambda_2 \Sigma \cdot m \delta y_i + \lambda_3 \Sigma \cdot m \delta z_i \\ & + \Lambda_1 \Sigma \cdot m \delta a_i + \Lambda_2 \Sigma \cdot m \delta b_i + \Lambda_3 \Sigma \cdot m \delta c_i; \end{aligned} \right\} \quad (L^1.)$$

and it resolves itself into the following separate expressions, in which the part V_i is considered as a function of the $6n + 1$ quantities $x_{i1} y_{i1} z_{i1} a_{i1} b_{i1} c_{i1} H_i$, of which, however, only $6n - 5$ are really independent:

first group,

$$\left. \begin{aligned} \frac{\delta V_i}{\delta x_{i1}} &= m_1 x'_{i1} + \lambda_1 m_1; \quad \dots \quad \frac{\delta V_i}{\delta x_{in}} = m_n x'_{in} + \lambda_1 m_n; \\ \frac{\delta V_i}{\delta y_{i1}} &= m_1 y'_{i1} + \lambda_2 m_1; \quad \dots \quad \frac{\delta V_i}{\delta y_{in}} = m_n y'_{in} + \lambda_2 m_n; \\ \frac{\delta V_i}{\delta z_{i1}} &= m_1 z'_{i1} + \lambda_3 m_1; \quad \dots \quad \frac{\delta V_i}{\delta z_{in}} = m_n z'_{in} + \lambda_3 m_n; \end{aligned} \right\} \quad (M^1.)$$

second group,

$$\left. \begin{aligned} \frac{\delta V_i}{\delta a_{i1}} &= -m_1 a'_{i1} + \Lambda_1 m_1; \quad \dots \quad \frac{\delta V_i}{\delta a_{in}} = -m_n a'_{in} + \Lambda_1 m_n; \\ \frac{\delta V_i}{\delta b_{i1}} &= -m_1 b'_{i1} + \Lambda_2 m_1; \quad \dots \quad \frac{\delta V_i}{\delta b_{in}} = -m_n b'_{in} + \Lambda_2 m_n; \\ \frac{\delta V_i}{\delta c_{i1}} &= -m_1 c'_{i1} + \Lambda_3 m_1; \quad \dots \quad \frac{\delta V_i}{\delta c_{in}} = -m_n c'_{in} + \Lambda_3 m_n; \end{aligned} \right\} \quad (N^1.)$$

and finally,

$$\frac{\delta V_i}{\delta H_i} = t. \quad \dots \dots \dots \dots \dots \dots \dots \quad (O^1.)$$

With respect to the six multipliers $\lambda_1 \lambda_2 \lambda_3 \Lambda_1 \Lambda_2 \Lambda_3$ which were introduced by the 3 final equations of condition $(45.)$, and by the 3 analogous initial equations of condition,

$$\Sigma \cdot m a_i = 0, \quad \Sigma \cdot m b_i = 0, \quad \Sigma \cdot m c_i = 0; \quad \dots \dots \dots \quad (55.)$$

we have, by differentiating these conditions,

$$\Sigma \cdot m x'_i = 0, \quad \Sigma \cdot m y'_i = 0, \quad \Sigma \cdot m z'_i = 0, \quad \dots \dots \quad (56.)$$

and

$$\Sigma \cdot m a'_i = 0, \quad \Sigma \cdot m b'_i = 0, \quad \Sigma \cdot m c'_i = 0; \quad \dots \dots \quad (57.)$$

and therefore

$$\lambda_1 = \frac{\Sigma \frac{\delta V_i}{\delta x_i}}{\Sigma m}, \quad \lambda_2 = \frac{\Sigma \frac{\delta V_i}{\delta y_i}}{\Sigma m}, \quad \lambda_3 = \frac{\Sigma \frac{\delta V_i}{\delta z_i}}{\Sigma m}, \quad \dots \dots \quad (58.)$$

and

$$\Lambda_1 = \frac{\Sigma \frac{\delta V_i}{\delta a_i}}{\Sigma m}, \quad \Lambda_2 = \frac{\Sigma \frac{\delta V_i}{\delta b_i}}{\Sigma m}, \quad \Lambda_3 = \frac{\Sigma \frac{\delta V_i}{\delta c_i}}{\Sigma m}. \quad \dots \dots \quad (59.)$$

11. As an example of the determination of these multipliers, we may suppose that the part V_p of the whole action V , has been expressed, before differentiation, as a function of H_i , and of these other $6n-6$ independent quantities

$$\left. \begin{aligned} x_1 - x_n &= \xi_1, & x_2 - x_n &= \xi_2, & \dots & x_{n-1} - x_n &= \xi_{n-1}, \\ y_1 - y_n &= \eta_1, & y_2 - y_n &= \eta_2, & \dots & y_{n-1} - y_n &= \eta_{n-1}, \\ z_1 - z_n &= \zeta_1, & z_2 - z_n &= \zeta_2, & \dots & z_{n-1} - z_n &= \zeta_{n-1}, \end{aligned} \right\} \quad \dots \quad (60.)$$

and

$$\left. \begin{aligned} a_1 - a_n &= \alpha_1, & a_2 - a_n &= \alpha_2, & \dots & a_{n-1} - a_n &= \alpha_{n-1}, \\ b_1 - b_n &= \beta_1, & b_2 - b_n &= \beta_2, & \dots & b_{n-1} - b_n &= \beta_{n-1}, \\ c_1 - c_n &= \gamma_1, & c_2 - c_n &= \gamma_2, & \dots & c_{n-1} - c_n &= \gamma_{n-1}, \end{aligned} \right\} \quad \dots \quad (61.)$$

that is, of the *differences* only of the *centrobaric* coordinates; or, in other words, as a function of the coordinates (initial and final) of $n-1$ points of the system, referred to the n^{th} point, as an internal or moveable origin: because the centrobaric coordinates $x_{q^i} y_{q^i} z_{q^i} a_{q^i} b_{q^i} c_{q^i}$ may themselves, by the equations of condition, be expressed as functions of these, namely,

$$x_i = \xi_i - \frac{\Sigma \cdot m \xi}{\Sigma m}, \quad y_i = \eta_i - \frac{\Sigma \cdot m \eta}{\Sigma m}, \quad z_i = \zeta_i - \frac{\Sigma \cdot m \zeta}{\Sigma m}, \quad \dots \quad (62.)$$

and in like manner,

$$a_i = \alpha_i - \frac{\Sigma \cdot m \alpha}{\Sigma m}, \quad b_i = \beta_i - \frac{\Sigma \cdot m \beta}{\Sigma m}, \quad c_i = \gamma_i - \frac{\Sigma \cdot m \gamma}{\Sigma m}; \quad \dots \quad (63.)$$

in which we are to observe, that the six quantities $\xi_n \eta_n \zeta_n \alpha_n \beta_n \gamma_n$ must be considered as separately vanishing. When V , has been thus expressed as a function of the centrobaric coordinates, involving their differences only, it will evidently satisfy the six partial differential equations,

$$\left. \begin{aligned} \Sigma \frac{\delta V_i}{\delta x_i} &= 0, & \Sigma \frac{\delta V_i}{\delta y_i} &= 0, & \Sigma \frac{\delta V_i}{\delta z_i} &= 0, \\ \Sigma \frac{\delta V_i}{\delta a_i} &= 0, & \Sigma \frac{\delta V_i}{\delta b_i} &= 0, & \Sigma \frac{\delta V_i}{\delta c_i} &= 0; \end{aligned} \right\} \quad \dots \quad (P1.)$$

after this preparation, therefore, of the function V_ν , the six multipliers determined by (58.) and (59.) will vanish, so that we shall have

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0, \dots \quad (64.)$$

and the groups (M¹) and (N¹) will reduce themselves to the two following :

$$\left. \begin{array}{l} \frac{\delta V_\nu}{\delta x_{i1}} = m_1 x'_{i1}; \frac{\delta V_\nu}{\delta x_{i2}} = m_2 x'_{i2}; \dots \frac{\delta V_\nu}{\delta x_{in}} = m_n x'_{in}; \\ \frac{\delta V_\nu}{\delta y_{i1}} = m_1 y'_{i1}; \frac{\delta V_\nu}{\delta y_{i2}} = m_2 y'_{i2}; \dots \frac{\delta V_\nu}{\delta y_{in}} = m_n y'_{in}; \\ \frac{\delta V_\nu}{\delta z_{i1}} = m_1 z'_{i1}; \frac{\delta V_\nu}{\delta z_{i2}} = m_2 z'_{i2}; \dots \frac{\delta V_\nu}{\delta z_{in}} = m_n z'_{in}; \end{array} \right\} \dots \quad (Q^1.)$$

and

$$\left. \begin{array}{l} \frac{\delta V_\nu}{\delta a_{i1}} = -m_1 a'_{i1}; \frac{\delta V_\nu}{\delta a_{i2}} = -m_2 a'_{i2}; \dots \frac{\delta V_\nu}{\delta a_{in}} = -m_n a'_{in}; \\ \frac{\delta V_\nu}{\delta b_{i1}} = -m_1 b'_{i1}; \frac{\delta V_\nu}{\delta b_{i2}} = -m_2 b'_{i2}; \dots \frac{\delta V_\nu}{\delta b_{in}} = -m_n b'_{in}; \\ \frac{\delta V_\nu}{\delta c_{i1}} = -m_1 c'_{i1}; \frac{\delta V_\nu}{\delta c_{i2}} = -m_2 c'_{i2}; \dots \frac{\delta V_\nu}{\delta c_{in}} = -m_n c'_{in}; \end{array} \right\} \dots \quad (R^1.)$$

analogous in all respects to the groups (C.) and (D.). We find, therefore, for the relative motion of a system about its own centre of gravity, equations of the same form as those which we had obtained before for the absolute motion of the same system of points in space. And we see that in investigating such relative motion only, it is useful to confine ourselves to the part V_ν of our whole characteristic function, that is, to the *relative action* of the system, or accumulated living force of the motion about the centre of gravity; and to consider this part as the *characteristic function* of such relative motion, in a sense analogous to that which has been already explained.

This relative action, or part V_ν , may, however, be otherwise expressed, and even in an infinite variety of ways, on account of the six equations of condition which connect the $6n$ centrobaric coordinates; and every different preparation of its form will give a different set of values for the six multipliers $\lambda_1 \lambda_2 \lambda_3 \Delta_1 \Delta_2 \Delta_3$. For example, we might eliminate, by a previous preparation, the six centrobaric coordinates of the point m_n from the expression of V_ν , so as to make this expression involve only the centrobaric coordinates of the other $n-1$ points of the system, and then we should have

$$\frac{\delta V_\nu}{\delta x_{in}} = 0, \frac{\delta V_\nu}{\delta y_{in}} = 0, \frac{\delta V_\nu}{\delta z_{in}} = 0, \frac{\delta V_\nu}{\delta a_{in}} = 0, \frac{\delta V_\nu}{\delta b_{in}} = 0, \frac{\delta V_\nu}{\delta c_{in}} = 0, \dots \quad (S^1.)$$

and therefore, by the six last equations of the groups (M¹) and (N¹), the multipliers would take the values

$$\lambda_1 = -x'_{in}, \lambda_2 = -y'_{in}, \lambda_3 = -z'_{in}, \Delta_1 = a'_{in}, \Delta_2 = b'_{in}, \Delta_3 = c'_{in}, \dots \quad (65.)$$

and would reduce, by (60.) and (61.), the preceding $6n-6$ equations of the same groups (M¹) and (N¹), to the forms

$$\left. \begin{aligned} \frac{\delta V_i}{\delta x_{i1}} &= m_1 \xi'_1, \frac{\delta V_i}{\delta x_{i2}} = m_2 \xi'_2, \dots, \frac{\delta V_i}{\delta x_{in-1}} = m_{n-1} \xi'_{n-1}, \\ \frac{\delta V_i}{\delta y_{i1}} &= m_1 \eta'_1, \frac{\delta V_i}{\delta y_{i2}} = m_2 \eta'_2, \dots, \frac{\delta V_i}{\delta y_{in-1}} = m_{n-1} \eta'_{n-1}, \\ \frac{\delta V_i}{\delta z_{i1}} &= m_1 \zeta'_1, \frac{\delta V_i}{\delta z_{i2}} = m_2 \zeta'_2, \dots, \frac{\delta V_i}{\delta z_{in-1}} = m_{n-1} \zeta'_{n-1}, \end{aligned} \right\} \quad \dots \quad (T^1)$$

and

$$\left. \begin{aligned} \frac{\delta V_i}{\delta a_{i1}} &= -m_1 \alpha'_1, \frac{\delta V_i}{\delta a_{i2}} = -m_2 \alpha'_2, \dots, \frac{\delta V_i}{\delta a_{in-1}} = -m_{n-1} \alpha'_{n-1}, \\ \frac{\delta V_i}{\delta b_{i1}} &= -m_1 \beta'_1, \frac{\delta V_i}{\delta b_{i2}} = -m_2 \beta'_2, \dots, \frac{\delta V_i}{\delta b_{in-1}} = -m_{n-1} \beta'_{n-1}, \\ \frac{\delta V_i}{\delta c_{i1}} &= -m_1 \gamma'_1, \frac{\delta V_i}{\delta c_{i2}} = -m_2 \gamma'_2, \dots, \frac{\delta V_i}{\delta c_{in-1}} = -m_{n-1} \gamma'_{n-1}. \end{aligned} \right\} \quad (U^1)$$

12. We might also express the relative action V_β , not as a function of the centrobaric, but of some other internal coordinates, or marks of relative position. We might, for instance, express it and its variation as functions of the $6n - 6$ independent internal coordinates $\xi \eta \zeta \alpha \beta \gamma$ already mentioned, and of their variations, defining these without any reference to the centre of gravity, by the equations

$$\left. \begin{aligned} \xi_i &= x_i - x_n, \eta_i = y_i - y_n, \zeta_i = z_i - z_n, \\ \alpha_i &= a_i - a_n, \beta_i = b_i - b_n, \gamma_i = c_i - c_n. \end{aligned} \right\} \quad \dots \quad (66)$$

For all such transformations of δV_i it is easy to establish a rule or law, which may be called the *law of varying relative action* (exactly analogous to the rule (B¹.)), namely, the following :

$$\delta V_i = \Sigma \cdot \left(\frac{\delta T_i}{\delta \eta'_i} \right) \delta \eta_i - \Sigma \cdot \left(\frac{\delta T_{i0}}{\delta e'_i} \right) \delta e_i + t \delta H_i + \Sigma \cdot \lambda_i \delta \varphi_i + \Sigma \cdot \Lambda_i \delta \Phi_i; \quad (V^1)$$

which implies that we are to express the half T_i of the relative living force of the system as a function of the rates of increase η'_i of any marks of relative position; and after taking its variation with respect to these rates, to change their variations to the variations of the marks of position themselves; then to subtract the initial from the final value of the result, and to add the variations of the final and initial functions $\varphi_i \Phi_i$, which enter into the equations of condition, if any, of the form $\varphi_i = 0, \Phi_i = 0$, (connecting the final and initial marks of relative position,) multiplied respectively by undetermined factors $\lambda_i \Lambda_i$; and lastly, to equate the whole result to $\delta V_i - t \delta H_i$, H_i being the quantity independent of the time in the equation (50.) of relative living force, and V_i being the relative action, of which we desired to express the variation. It is not necessary to dwell here on the demonstration of this new rule (V¹.), which may easily be deduced from the principles already laid down; or by the calculus of variations from the law of relative living force, combined with the differential equations of the second order of relative motion.

But to give an example of its application, let us resume the problem already mentioned, namely to express δV , by means of the $6n - 5$ independent variations $\delta \xi_i \delta \eta_i \delta \zeta_i \delta \alpha_i \delta \beta_i \delta \gamma_i \delta H_i$. For this purpose we shall employ a known transformation of the relative living force $2T_i$, multiplied by the sum of the masses of the system, namely the following :

$$2T_i \Sigma m = \Sigma . m_i m_k \{ (x'_i - x'_k)^2 + (y'_i - y'_k)^2 + (z'_i - z'_k)^2 \} : \quad \dots \quad (67.)$$

the sign of summation Σ extending, in the second member, to all the combinations of points two by two, which can be formed without repetition. This transformation gives, by (66.),

$$2T_i \Sigma m = m_n \Sigma_i . m (\xi'^2 + \eta'^2 + \zeta'^2) + \Sigma_i . m_i m_k \{ (\xi'_i - \xi'_k)^2 + (\eta'_i - \eta'_k)^2 + (\zeta'_i - \zeta'_k)^2 \} ; \quad \dots \quad (68.)$$

the sign of summation Σ_i extending only to the first $n - 1$ points of the system. Applying, therefore, our general rule or law of varying relative action, and observing that the $6n - 6$ internal coordinates $\xi \eta \zeta \alpha \beta \gamma$ are independent, we find the following new expression :

$$\left. \begin{aligned} \delta V_i &= t \delta H_i + \frac{m_n}{\Sigma m} \cdot \Sigma_i . m (\xi' \delta \xi - \alpha' \delta \alpha + \eta' \delta \eta - \beta' \delta \beta + \zeta' \delta \zeta - \gamma' \delta \gamma) \\ &+ \frac{1}{\Sigma m} \cdot \Sigma_i . m_i m_k \{ (\xi'_i - \xi'_k) (\delta \xi_i - \delta \xi_k) + (\eta'_i - \eta'_k) (\delta \eta_i - \delta \eta_k) + (\zeta'_i - \zeta'_k) (\delta \zeta_i - \delta \zeta_k) \} \\ &- \frac{1}{\Sigma m} \cdot \Sigma_i . m_i m_k \{ (\alpha'_i - \alpha'_k) (\delta \alpha_i - \delta \alpha_k) + (\beta'_i - \beta'_k) (\delta \beta_i - \delta \beta_k) + (\gamma'_i - \gamma'_k) (\delta \gamma_i - \delta \gamma_k) \} : \end{aligned} \right\} (W^1.)$$

which gives, besides the equation (O^1) , the following groups :

$$\left. \begin{aligned} \frac{\delta V_i}{\delta \xi_i} &= \frac{m_i}{\Sigma m} \cdot \Sigma . m (\xi'_i - \xi') = m_i \left(\xi'_i - \frac{\Sigma_i m \xi'}{\Sigma m} \right), \\ \frac{\delta V_i}{\delta \eta_i} &= \frac{m_i}{\Sigma m} \cdot \Sigma . m (\eta'_i - \eta') = m_i \left(\eta'_i - \frac{\Sigma_i m \eta'}{\Sigma m} \right), \\ \frac{\delta V_i}{\delta \zeta_i} &= \frac{m_i}{\Sigma m} \cdot \Sigma . m (\zeta'_i - \zeta') = m_i \left(\zeta'_i - \frac{\Sigma_i m \zeta'}{\Sigma m} \right), \end{aligned} \right\} \quad \dots \quad (X^1.)$$

and

$$\left. \begin{aligned} \frac{\delta V_i}{\delta \alpha_i} &= - \frac{m_i}{\Sigma m} \cdot \Sigma . m (\alpha'_i - \alpha') = - m_i \left(\alpha'_i - \frac{\Sigma_i m \alpha'}{\Sigma m} \right), \\ \frac{\delta V_i}{\delta \beta_i} &= - \frac{m_i}{\Sigma m} \cdot \Sigma . m (\beta'_i - \beta') = - m_i \left(\beta'_i - \frac{\Sigma_i m \beta'}{\Sigma m} \right), \\ \frac{\delta V_i}{\delta \gamma_i} &= - \frac{m_i}{\Sigma m} \cdot \Sigma . m (\gamma'_i - \gamma') = - m_i \left(\gamma'_i - \frac{\Sigma_i m \gamma'}{\Sigma m} \right); \end{aligned} \right\} \quad \dots \quad (Y^1.)$$

results which may be thus summed up :

$$\left. \begin{aligned} \delta V_i &= t \delta H_i + \sum_i m (\xi' \delta \xi - \alpha' \delta \alpha + \eta' \delta \eta - \beta' \delta \beta + \zeta' \delta \zeta - \gamma' \delta \gamma) \\ &\quad - \frac{1}{\sum m} (\sum_i m \xi' \cdot \sum_i m \delta \xi + \sum_i m \eta' \cdot \sum_i m \delta \eta + \sum_i m \beta' \cdot \sum_i m \delta \beta + \sum_i m \gamma' \cdot \sum_i m \delta \gamma) \\ &\quad + \frac{1}{\sum m} (\sum_i m \alpha' \cdot \sum_i m \delta \alpha + \sum_i m \beta' \cdot \sum_i m \delta \beta + \sum_i m \gamma' \cdot \sum_i m \delta \gamma), \end{aligned} \right\} . \quad (Z^1.)$$

and might have been otherwise deduced by our rule, from this other known transformation of T_i ,

$$T_i = \frac{1}{2} \sum_i m (\xi'^2 + \eta'^2 + \zeta'^2) - \frac{(\sum_i m \xi')^2 + (\sum_i m \eta')^2 + (\sum_i m \zeta')^2}{2 \sum m}. \quad \dots \quad (69.)$$

And to obtain, with any set of internal or relative marks of position, the two partial differential equations which the characteristic function V_i of relative motion must satisfy, and which offer (as we shall find) the chief means of discovering its form, namely, the equations analogous to those marked (F.) and (G.), we have only to eliminate the rates of increase of the marks of position of the system, which determine the final and initial components of the relative velocities of its points, by the law of varying relative action, from the final and initial expressions of the law of relative living force; namely, from the following equations:

$$T_i = U + H_i, \quad \dots \quad (50.)$$

and

$$T_{i0} = U_0 + H_{i0}, \quad \dots \quad (70.)$$

The law of areas, or the property respecting rotation which was expressed by the partial differential equations (P.), will also always admit of being expressed in relative coordinates, and will assist in discovering the form of the characteristic function V_i ; by showing that this function involves only such internal coordinates (in number $6n - 9$) as do not alter by any common rotation of all points final and initial, round the centre of gravity, or round any other internal origin; that origin being treated as fixed, and the quantity H_i as constant, in determining the effects of this rotation. The general problem of dynamics, respecting the motions of a free system of n points attracting or repelling one another, is therefore reduced, in the last analysis, by the method of the present essay, to the research and differentiation of a function V_i , depending on $6n - 9$ internal or relative coordinates, and on the quantity H_i , and satisfying a pair of partial differential equations of the first order and second degree; in integrating which equations, we are to observe, that at the assumed origin of the motion, namely at the moment when $t = 0$, the final or variable coordinates are equal to their initial values, and the partial differential coefficient $\frac{\delta V_i}{\delta H_i}$ vanishes; and, that at a moment infinitely little distant, the differential alterations of the coordinates have ratios connected with the other partial differential coefficients of the characteristic function V_i , by the law of varying relative action. It may be here observed, that,

although the consideration of the point, called usually the centre of gravity, is very simply suggested by the process of the tenth number, yet this internal centre is even more simply indicated by our early corollaries from the law of varying action; which show that the components of relative final velocities, in any system of attracting or repelling points, may be expressed by the differences of quantities of the form $\frac{1}{m} \frac{\delta V}{\delta x}$, $\frac{1}{m} \frac{\delta V}{\delta y}$, $\frac{1}{m} \frac{\delta V}{\delta z}$: and therefore that in calculating these relative velocities, it is advantageous to introduce the final sums $\Sigma m x$, $\Sigma m y$, $\Sigma m z$, and, for an analogous reason, the initial sums $\Sigma m a$, $\Sigma m b$, $\Sigma m c$, among the marks of the extreme positions of the system, in the expression of the characteristic function V ; because, in differentiating that expression for the calculation of relative velocities, those sums may be treated as constant.

On Systems of two Points, in general; Characteristic Function of the motion of any Binary System.

13. To illustrate the foregoing principles, which extend to any free system of points, however numerous, attracting or repelling one another, let us now consider, in particular, a system of two such points. For such a system, the known *force-function* U becomes, by (2.),

$$U = m_1 m_2 f(r), \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (71.)$$

r being the mutual distance

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}, \quad \dots \quad \dots \quad \dots \quad (72.)$$

between the two points m_1 , m_2 , and $f(r)$ being a function of this distance such that its derivative or differential coefficient $f'(r)$ expresses the law of their repulsion or attraction, according as it is positive or negative. The known differential equations of motion, of the second order, are now, by (1.), comprised in the following formula:

$$m_1 (x''_1 \delta x_1 + y''_1 \delta y_1 + z''_1 \delta z_1) + m_2 (x''_2 \delta x_2 + y''_2 \delta y_2 + z''_2 \delta z_2) = m_1 m_2 \delta f(r); \quad \dots \quad (73.)$$

they are therefore, separately,

$$\left. \begin{aligned} x''_1 &= m_2 \frac{\delta f(r)}{\delta x_1}, & y''_1 &= m_2 \frac{\delta f(r)}{\delta y_1}, & z''_1 &= m_2 \frac{\delta f(r)}{\delta z_1}, \\ x''_2 &= m_1 \frac{\delta f(r)}{\delta x_2}, & y''_2 &= m_1 \frac{\delta f(r)}{\delta y_2}, & z''_2 &= m_1 \frac{\delta f(r)}{\delta z_2}. \end{aligned} \right\} \quad \dots \quad \dots \quad (74.)$$

The problem of integrating these equations consists in proposing to assign, by their means, six relations between the time t , the masses m_1 , m_2 , the six varying coordinates x_1 , y_1 , z_1 , x_2 , y_2 , z_2 , and their initial values and initial rates of increase a_1 , b_1 , c_1 , a_2 , b_2 , c_2 , d'_1 , b'_1 , c'_1 , d'_2 , b'_2 , c'_2 . If we knew these six final integrals, and combined them with the initial form of the law of living force, or of the known intermediate integral

$$\frac{1}{2} m_1 (x'_1{}^2 + y'_1{}^2 + z'_1{}^2) + \frac{1}{2} m_2 (x'_2{}^2 + y'_2{}^2 + z'_2{}^2) = m_1 m_2 f(r) + H; \quad \dots \quad (75.)$$

that is, with the following formula,

$$\frac{1}{2}m_1(a'_1{}^2 + b'_1{}^2 + c'_1{}^2) + \frac{1}{2}m_2(a'_2{}^2 + b'_2{}^2 + c'_2{}^2) = m_1m_2f(r_0) + H, \quad \dots \quad (76.)$$

in which r_0 is the initial distance

$$r_0 = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2}, \quad \dots \quad (77.)$$

and H is a constant quantity, introduced by integration; we could, by the combination of these seven relations, determine the time t , and the six initial components of velocity $a'_1 b'_1 c'_1 a'_2 b'_2 c'_2$, as functions of the twelve final and initial coordinates $x_1 y_1 z_1 x_2 y_2 z_2 a_1 b_1 c_1 a_2 b_2 c_2$, and of the quantity H , (involving also the masses:) we could therefore determine whatever else depends on the manner and time of motion of this system of two points, as a function of the same extreme coordinates and of the same quantity H . In particular, we could determine the action, or accumulated living force of the system, namely,

$$V = m_1 \int_0^t (x'_1{}^2 + y'_1{}^2 + z'_1{}^2) dt + m_2 \int_0^t (x'_2{}^2 + y'_2{}^2 + z'_2{}^2) dt, \quad \dots \quad (A^2.)$$

as a function of those thirteen quantities $x_1 y_1 z_1 x_2 y_2 z_2 a_1 b_1 c_1 a_2 b_2 c_2 H$: and might then calculate the variation of this function,

$$\left. \begin{aligned} \delta V &= \frac{\delta V}{\delta x_1} \delta x_1 + \frac{\delta V}{\delta y_1} \delta y_1 + \frac{\delta V}{\delta z_1} \delta z_1 + \frac{\delta V}{\delta x_2} \delta x_2 + \frac{\delta V}{\delta y_2} \delta y_2 + \frac{\delta V}{\delta z_2} \delta z_2 \\ &+ \frac{\delta V}{\delta a_1} \delta a_1 + \frac{\delta V}{\delta b_1} \delta b_1 + \frac{\delta V}{\delta c_1} \delta c_1 + \frac{\delta V}{\delta a_2} \delta a_2 + \frac{\delta V}{\delta b_2} \delta b_2 + \frac{\delta V}{\delta c_2} \delta c_2 \\ &+ \frac{\delta V}{\delta H} \delta H. \end{aligned} \right\} \quad (B^2.)$$

But the essence of our method consists in *forming previously the expression of this variation, by our law of varying action*, namely,

$$\left. \begin{aligned} \delta V &= m_1(x'_1 \delta x_1 - a'_1 \delta a_1 + y'_1 \delta y_1 - b'_1 \delta b_1 + z'_1 \delta z_1 - c'_1 \delta c_1) \\ &+ m_2(x'_2 \delta x_2 - a'_2 \delta a_2 + y'_2 \delta y_2 - b'_2 \delta b_2 + z'_2 \delta z_2 - c'_2 \delta c_2) \\ &+ t \delta H; \end{aligned} \right\} \quad (C^2.)$$

and in *considering V as a characteristic function of the motion*, from the form of which may be deduced all the intermediate and all the final integrals of the known differential equations, by resolving the expression $(C^2.)$ into the following separate groups, (included in $(C.)$ and $(D.)$),

$$\left. \begin{aligned} \frac{\delta V}{\delta x_1} &= m_1 x'_1, \quad \frac{\delta V}{\delta y_1} = m_1 y'_1, \quad \frac{\delta V}{\delta z_1} = m_1 z'_1, \\ \frac{\delta V}{\delta x_2} &= m_2 x'_2, \quad \frac{\delta V}{\delta y_2} = m_2 y'_2, \quad \frac{\delta V}{\delta z_2} = m_2 z'_2; \end{aligned} \right\} \quad \dots \quad (D^2.)$$

and

$$\left. \begin{aligned} \frac{\delta V}{\delta a_1} &= -m_1 a'_1, \quad \frac{\delta V}{\delta b_1} = -m_1 b'_1, \quad \frac{\delta V}{\delta c_1} = -m_1 c'_1, \\ \frac{\delta V}{\delta a_2} &= -m_2 a'_2, \quad \frac{\delta V}{\delta b_2} = -m_2 b'_2, \quad \frac{\delta V}{\delta c_2} = -m_2 c'_2; \end{aligned} \right\} \quad \dots \quad (E^2.)$$

besides this other equation, which had occurred before,

$$\frac{\delta V}{\delta H} = t. \quad \dots \dots \dots \dots \dots \dots \quad (E.)$$

By this new method, the difficulty of integrating the six known equations of motion of the second order (74.), is reduced to the search and differentiation of a single function V ; and to find the form of this function, we are to employ the following pair of partial differential equations of the first order:

$$\begin{aligned} & \frac{1}{2m_1} \left\{ \left(\frac{\delta V}{\delta x_1} \right)^2 + \left(\frac{\delta V}{\delta y_1} \right)^2 + \left(\frac{\delta V}{\delta z_1} \right)^2 \right\} + \frac{1}{2m_2} \left\{ \left(\frac{\delta V}{\delta x_2} \right)^2 + \left(\frac{\delta V}{\delta y_2} \right)^2 + \left(\frac{\delta V}{\delta z_2} \right)^2 \right\} \\ & = m_1 m_2 f(r) + H, \quad \dots \dots \dots \dots \dots \dots \quad (F^2.) \\ & \frac{1}{2m_1} \left\{ \left(\frac{\delta V}{\delta a_1} \right)^2 + \left(\frac{\delta V}{\delta b_1} \right)^2 + \left(\frac{\delta V}{\delta c_1} \right)^2 \right\} + \frac{1}{2m_2} \left\{ \left(\frac{\delta V}{\delta a_2} \right)^2 + \left(\frac{\delta V}{\delta b_2} \right)^2 + \left(\frac{\delta V}{\delta c_2} \right)^2 \right\} \\ & = m_1 m_2 f(r_0) + H, \quad \dots \dots \dots \dots \dots \dots \quad (G^2.) \end{aligned}$$

combined with some simple considerations. And it easily results from the principles already laid down, that the integral of this pair of equations, adapted to the present question, is

$$\begin{aligned} V &= \sqrt{(x_u - a_u)^2 + (y_u - b_u)^2 + (z_u - c_u)^2} \cdot \sqrt{2H_u(m_1 + m_2)} \\ &+ \frac{m_1 m_2}{m_1 + m_2} (h \Omega + \int_{r_0}^r \varrho dr); \quad \dots \dots \dots \dots \dots \dots \quad (H^2.) \end{aligned}$$

in which $x_u, y_u, z_u, a_u, b_u, c_u$, denote the coordinates, final and initial, of the centre of gravity of the system,

$$\left. \begin{aligned} x_u &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad y_u = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, \quad z_u = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}, \\ a_u &= \frac{m_1 a_1 + m_2 a_2}{m_1 + m_2}, \quad b_u = \frac{m_1 b_1 + m_2 b_2}{m_1 + m_2}, \quad c_u = \frac{m_1 c_1 + m_2 c_2}{m_1 + m_2}, \end{aligned} \right\} \quad \dots \dots \dots \quad (78.)$$

and Ω is the angle between the final and initial distances r, r_0 : we have also put for abridgement

$$\varrho = \pm \sqrt{2(m_1 + m_2) \left(f(r) + \frac{H_i}{m_1 m_2} \right) - \frac{h^2}{r^2}}, \quad \dots \dots \dots \dots \dots \quad (79.)$$

the upper or the lower sign to be used, according as the distance r is increasing or decreasing; and have introduced three auxiliary quantities h, H_i, H_u , to be determined by this condition,

$$0 = \Omega + \int_{r_0}^r \frac{\delta \varrho}{\delta h} dr, \quad \dots \dots \dots \dots \dots \dots \quad (I^2.)$$

combined with the two following,

$$\left. \begin{aligned} \frac{m_1 m_2}{m_1 + m_2} \int_{r_0}^r \frac{\delta \varrho}{\delta H_i} dr &= \sqrt{(x_u - a_u)^2 + (y_u - b_u)^2 + (z_u - c_u)^2} \cdot \sqrt{\frac{m_1 + m_2}{2H_u}}, \\ H_i + H_u &= H; \end{aligned} \right\} \quad (K^2.)$$

which auxiliary quantities, although in one view they are functions of the twelve extreme coordinates, are yet to be treated as constant in calculating the three definite integrals, or limits of sums of numerous small elements,

$$\int_{r_0}^r \varrho \, dr, \int_{r_0}^r \frac{\delta \varrho}{\delta h} \, dr, \int_{r_0}^r \frac{\delta \varrho}{\delta H_i} \, dr.$$

The form (H^2) , for the *characteristic function of a binary system*, may be regarded as a central or radical relation, which includes the whole theory of the motion of such a system; so that all the details of this motion may be deduced from it by the application of our general method. But because the theory of binary systems has been brought to great perfection already, by the labours of former writers, it may suffice to give briefly here a few instances of such deduction.

14. The form $(H^2)_s$, for the characteristic function of a binary system involves explicitly, when g is changed to its value (79.), the twelve quantities $x_u y_u z_u a_u b_u c_u$, $r r_0 \propto h H_1 H_{12}$ (besides the masses $m_1 m_2$ which are always considered as given); its variation may therefore be thus expressed:

$$\delta V = \left. \begin{aligned} & \frac{\delta V}{\delta x_u} \delta x_u + \frac{\delta V}{\delta y_u} \delta y_u + \frac{\delta V}{\delta z_u} \delta z_u + \frac{\delta V}{\delta a_u} \delta a_u + \frac{\delta V}{\delta b_u} \delta b_u + \frac{\delta V}{\delta c_u} \delta c_u \\ & + \frac{\delta V}{\delta r} \delta r + \frac{\delta V}{\delta r_0} \delta r_0 + \frac{\delta V}{\delta S} \delta S + \frac{\delta V}{\delta h} \delta h + \frac{\delta V}{\delta H_i} \delta H_i + \frac{\delta V}{\delta H_u} \delta H_u \end{aligned} \right\} . \quad (L^2)$$

In this expression, if we put for abridgement

$$\lambda = \sqrt{\frac{\omega H_u(m_1 + m_2)}{(x_u - a_u)^2 + (y_u - b_u)^2 + (z_u - c_u)^2}}, \quad \dots \quad (80.)$$

we shall have

$$\left. \begin{aligned} \frac{\delta V}{\delta x_u} &= \lambda (x_u - a_u), \quad \frac{\delta V}{\delta y_u} = \lambda (y_u - b_u), \quad \frac{\delta V}{\delta z_u} = \lambda (z_u - c_u), \\ \frac{\delta V}{\delta a_u} &= \lambda (a_u - x_u), \quad \frac{\delta V}{\delta b_u} = \lambda (b_u - y_u), \quad \frac{\delta V}{\delta c_u} = \lambda (c_u - z_u); \end{aligned} \right\} \quad \ldots \quad \ldots \quad \ldots \quad (M^2).$$

and if we put

$$z_0 = \pm \sqrt{2(m_1 + m_2) \left(f(r_0) + \frac{H_1}{m_1 m_2} \right) - \frac{\hbar^2}{r_0^2}}, \quad \dots \dots \dots \dots \quad (81.)$$

the sign of the radical being determined by the same rule as that of ϱ , we shall have

$$\frac{\delta V}{\delta r} = \frac{m_1 m_2 \varrho}{m_1 + m_2}, \quad \frac{\delta V}{\delta r_0} = -\frac{m_1 m_2 \varrho_0}{m_1 + m_2}, \quad \frac{\delta V}{\delta \mathfrak{H}} = \frac{m_1 m_2 h}{m_1 + m_2}; \quad \dots \quad (N^2.)$$

besides, by the equations of condition (I².), (K².), we have

$$\frac{\delta V}{\delta h} = 0, \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (O^2.)$$

and

$$\frac{\delta V}{\delta H_u} = \frac{\delta V}{\delta H_i} = \int_{r_0}^r \frac{dr}{\varrho}, \quad \delta H_i + \delta H_u = \delta H. \quad \dots \quad (P^2.)$$

The expression (L²) may therefore be thus transformed :

$$\delta V = \lambda \{(x_u - a_u)(\delta x_u - \delta a_u) + (y_u - b_u)(\delta y_u - \delta b_u) + (z_u - c_u)(\delta z_u - \delta c_u)\} + \frac{m_1 m_2}{m_1 + m_2} (\varepsilon \delta r - \varepsilon_0 \delta r_0 + h \delta \mathcal{H}) + \int_{r_0}^r \frac{dr}{g} \cdot \delta H; \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (Q^2.)$$

and may be resolved by our general method into twelve separate expressions for the final and initial components of velocities, namely,

$$\left. \begin{aligned} x'_1 &= \frac{1}{m_1} \frac{\delta V}{\delta x_1} = \frac{\lambda}{m_1 + m_2} (x_u - a_u) + \frac{m_2}{m_1 + m_2} \left(\varepsilon \frac{\delta r}{\delta x_1} + h \frac{\delta \mathcal{H}}{\delta x_1} \right), \\ y'_1 &= \frac{1}{m_1} \frac{\delta V}{\delta y_1} = \frac{\lambda}{m_1 + m_2} (y_u - b_u) + \frac{m_2}{m_1 + m_2} \left(\varepsilon \frac{\delta r}{\delta y_1} + h \frac{\delta \mathcal{H}}{\delta y_1} \right), \\ z'_1 &= \frac{1}{m_1} \frac{\delta V}{\delta z_1} = \frac{\lambda}{m_1 + m_2} (z_u - c_u) + \frac{m_2}{m_1 + m_2} \left(\varepsilon \frac{\delta r}{\delta z_1} + h \frac{\delta \mathcal{H}}{\delta z_1} \right), \\ x'_2 &= \frac{1}{m_2} \frac{\delta V}{\delta x_2} = \frac{\lambda}{m_1 + m_2} (x_u - a_u) + \frac{m_1}{m_1 + m_2} \left(\varepsilon \frac{\delta r}{\delta x_2} + h \frac{\delta \mathcal{H}}{\delta x_2} \right), \\ y'_2 &= \frac{1}{m_2} \frac{\delta V}{\delta y_2} = \frac{\lambda}{m_1 + m_2} (y_u - b_u) + \frac{m_1}{m_1 + m_2} \left(\varepsilon \frac{\delta r}{\delta y_2} + h \frac{\delta \mathcal{H}}{\delta y_2} \right), \\ z'_2 &= \frac{1}{m_2} \frac{\delta V}{\delta z_2} = \frac{\lambda}{m_1 + m_2} (z_u - c_u) + \frac{m_1}{m_1 + m_2} \left(\varepsilon \frac{\delta r}{\delta z_2} + h \frac{\delta \mathcal{H}}{\delta z_2} \right), \end{aligned} \right\} \quad \dots \quad (R^2.)$$

and

$$\left. \begin{aligned} a'_1 &= \frac{-1}{m_1} \frac{\delta V}{\delta a_1} = \frac{\lambda}{m_1 + m_2} (x_u - a_u) + \frac{m_2}{m_1 + m_2} \left(\varepsilon_0 \frac{\delta r_0}{\delta a_1} - h \frac{\delta \mathcal{H}}{\delta a_1} \right), \\ b'_1 &= \frac{-1}{m_1} \frac{\delta V}{\delta b_1} = \frac{\lambda}{m_1 + m_2} (y_u - b_u) + \frac{m_2}{m_1 + m_2} \left(\varepsilon_0 \frac{\delta r_0}{\delta b_1} - h \frac{\delta \mathcal{H}}{\delta b_1} \right), \\ c'_1 &= \frac{-1}{m_1} \frac{\delta V}{\delta c_1} = \frac{\lambda}{m_1 + m_2} (z_u - c_u) + \frac{m_2}{m_1 + m_2} \left(\varepsilon_0 \frac{\delta r_0}{\delta c_1} - h \frac{\delta \mathcal{H}}{\delta c_1} \right), \\ a'_2 &= \frac{-1}{m_2} \frac{\delta V}{\delta a_2} = \frac{\lambda}{m_1 + m_2} (x_u - a_u) + \frac{m_1}{m_1 + m_2} \left(\varepsilon_0 \frac{\delta r_0}{\delta a_2} - h \frac{\delta \mathcal{H}}{\delta a_2} \right), \\ b'_2 &= \frac{-1}{m_2} \frac{\delta V}{\delta b_2} = \frac{\lambda}{m_1 + m_2} (y_u - b_u) + \frac{m_1}{m_1 + m_2} \left(\varepsilon_0 \frac{\delta r_0}{\delta b_2} - h \frac{\delta \mathcal{H}}{\delta b_2} \right), \\ c'_2 &= \frac{-1}{m_2} \frac{\delta V}{\delta c_2} = \frac{\lambda}{m_1 + m_2} (z_u - c_u) + \frac{m_1}{m_1 + m_2} \left(\varepsilon_0 \frac{\delta r_0}{\delta c_2} - h \frac{\delta \mathcal{H}}{\delta c_2} \right); \end{aligned} \right\} \quad \dots \quad (S^2.)$$

besides the following expression for the time of motion of the system :

$$t = \frac{\delta V}{\delta H} = \int_{r_0}^r \frac{dr}{g}, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (T^2.)$$

which gives by (K²), and by (79.), (80.),

$$t = \frac{m_1 + m_2}{\lambda} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (U^2.)$$

The six equations (R²) give the six intermediate integrals, and the six equations (S²) give the six final integrals of the six known differential equations of motion (74.) for any binary system, if we eliminate or determine the three auxiliary quantities

h, H_x, H_y , by the three conditions (I².) (T².) (U²). Thus, if we observe that the distances r, r_0 , and the included angle \S , depend only on relative coordinates, which may be thus denoted,

$$\left. \begin{aligned} x_1 - x_2 &= \xi, & y_1 - y_2 &= \eta, & z_1 - z_2 &= \zeta, \\ a_1 - a_2 &= \alpha, & b_1 - b_2 &= \beta, & c_1 - c_2 &= \gamma, \end{aligned} \right\} \quad \dots \dots \dots \quad (82.)$$

we obtain by easy combinations the three following intermediate integrals for the centre of gravity of the system :

$$x'_u t = x_u - a_u, \quad y'_u t = y_u - b_u, \quad z'_u t = z_u - c_u, \quad \dots \dots \dots \quad (83.)$$

and the three following final integrals,

$$a'_u t = x_u - a_u, \quad b'_u t = y_u - b_u, \quad c'_u t = z_u - c_u, \quad \dots \dots \dots \quad (84.)$$

expressing the well-known law of the rectilinear and uniform motion of that centre. We obtain also the three following intermediate integrals for the relative motion of one point of the system about the other :

$$\left. \begin{aligned} \xi' &= \xi \frac{\partial r}{\partial \xi} + h \frac{\partial \S}{\partial \xi}, \\ \eta' &= \xi \frac{\partial r}{\partial \eta} + h \frac{\partial \S}{\partial \eta}, \\ \zeta' &= \xi \frac{\partial r}{\partial \zeta} + h \frac{\partial \S}{\partial \zeta}, \end{aligned} \right\} \quad \dots \dots \dots \quad (85.)$$

and the three following final integrals,

$$\left. \begin{aligned} \alpha' &= \xi_0 \frac{\partial r_0}{\partial \alpha} - h \frac{\partial \S}{\partial \alpha}, \\ \beta' &= \xi_0 \frac{\partial r_0}{\partial \beta} - h \frac{\partial \S}{\partial \beta}, \\ \gamma' &= \xi_0 \frac{\partial r_0}{\partial \gamma} - h \frac{\partial \S}{\partial \gamma}; \end{aligned} \right\} \quad \dots \dots \dots \quad (86.)$$

in which the auxiliary quantities h, H_x, H_y , are to be determined by (I².) (T².), and in which the dependence of r, r_0, \S , on $\xi, \eta, \zeta, \alpha, \beta, \gamma$, is expressed by the following equations :

$$\left. \begin{aligned} r &= \sqrt{\xi^2 + \eta^2 + \zeta^2}, & r_0 &= \sqrt{\alpha^2 + \beta^2 + \gamma^2}, \\ r r_0 \cos \S &= \xi \alpha + \eta \beta + \zeta \gamma. \end{aligned} \right\} \quad \dots \dots \dots \quad (87.)$$

If then we put, for abridgement,

$$A = \frac{\xi}{r} + \frac{h}{r^2 \tan \S}, \quad B = \frac{h}{r r_0 \sin \S}, \quad C = \frac{-\xi_0}{r_0} + \frac{h}{r_0^2 \tan \S}, \quad \dots \dots \dots \quad (88.)$$

we shall have these three intermediate integrals,

$$\xi' = A \xi - B \alpha, \quad \eta' = A \eta - B \beta, \quad \zeta' = A \zeta - B \gamma, \quad \dots \dots \dots \quad (89.)$$

and these three final integrals,

$$\alpha' = B \xi - C \alpha, \quad \beta' = B \eta - C \beta, \quad \gamma' = B \zeta - C \gamma, \quad \dots \dots \dots \quad (90.)$$

of the equations of relative motion. These integrals give,

$$\left. \begin{array}{l} \xi \eta' - \eta \xi' = \alpha \beta' - \beta \alpha' = B (\alpha \eta - \beta \xi), \\ \eta \xi' - \xi \eta' = \beta \gamma' - \gamma \beta' = B (\beta \xi - \gamma \eta), \\ \xi \xi' - \eta \eta' = \gamma \alpha' - \alpha \gamma' = B (\gamma \xi - \alpha \eta), \end{array} \right\} \quad \dots \dots \dots \dots \dots \dots \quad (91.)$$

and

$$\xi (\alpha \beta' - \beta \alpha') + \xi (\beta \gamma' - \gamma \beta') + \eta (\gamma \alpha' - \alpha \gamma') = 0; \quad \dots \dots \dots \dots \dots \dots \quad (92.)$$

they contain therefore the known law of equable description of areas, and the law of a plane relative orbit. If we take for simplicity this plane for the plane $\xi \eta$, the quantities $\xi \xi' \gamma \gamma'$ will vanish; and we may put,

$$\left. \begin{array}{l} \xi = r \cos \theta, \eta = r \sin \theta, \xi' = 0, \\ \alpha = r_0 \cos \theta_0, \beta = r_0 \sin \theta_0, \gamma = 0, \end{array} \right\} \quad \dots \dots \dots \dots \dots \dots \quad (93.)$$

and

$$\left. \begin{array}{l} \xi' = r' \cos \theta - \theta' r \sin \theta, \eta' = r' \sin \theta + \theta' r \cos \theta, \xi' = 0, \\ \alpha' = r'_0 \cos \theta_0 - \theta'_0 r_0 \sin \theta_0, \beta' = r'_0 \sin \theta_0 + \theta'_0 r_0 \cos \theta_0, \gamma' = 0, \end{array} \right\} \quad \dots \dots \quad (94.)$$

the angles $\theta \theta_0$ being counted from some fixed line in the plane, and being such that their difference

$$\theta - \theta_0 = \Omega. \quad \dots \dots \dots \dots \dots \dots \quad (95.)$$

These values give

$$\xi \eta' - \eta \xi' = r^2 \theta', \alpha \beta' - \beta \alpha' = r_0^2 \theta'_0, \alpha \eta - \beta \xi = r r_0 \sin \Omega, \quad \dots \dots \quad (96.)$$

and therefore, by (88.) and (91),

$$r^2 \theta' = r_0^2 \theta'_0 = h; \quad \dots \dots \dots \dots \dots \dots \quad (97.)$$

the quantity $\frac{1}{2} h$ is therefore the constant areal velocity in the relative motion of the system; a result which is easily seen to be independent of the directions of the three rectangular coordinates. The same values, (93.), (94.), give

$$\left. \begin{array}{l} \xi \cos \theta + \eta \sin \theta = r, \xi' \cos \theta + \eta' \sin \theta = r', \alpha \cos \theta + \beta \sin \theta = r_0 \cos \Omega, \\ \alpha \cos \theta_0 + \beta \sin \theta_0 = r_0, \alpha' \cos \theta_0 + \beta' \sin \theta_0 = r'_0, \xi \cos \theta_0 + \eta \sin \theta_0 = r \cos \Omega, \end{array} \right\} \quad (98.)$$

and therefore, by the intermediate and final integrals, (89.), (90.),

$$r' = \xi, r'_0 = \xi_0; \quad \dots \dots \dots \dots \dots \dots \quad (99.)$$

results which evidently agree with the condition (T²), and which give by (79.) and (81.), for all directions of coordinates,

$$\left. \begin{array}{l} r'^2 + \frac{h^2}{r^2} - 2 (m_1 + m_2) f(r) = \\ r'^2_0 + \frac{h^2}{r_0^2} - 2 (m_1 + m_2) f(r_0) = 2 H, \left(\frac{1}{m_1} + \frac{1}{m_2} \right); \end{array} \right\} \quad \dots \dots \dots \quad (100.)$$

the other auxiliary quantity H , is therefore also a constant, independent of the time, and enters as such into the constant part in the expression for $\left(r'^2 + \frac{h^2}{r^2} \right)$ the square of the relative velocity. The equation of condition (I²), connecting these two con-

stants h , H_i , with the extreme lengths of the radius vector r , and with the angle \S described by this radius in revolving from its initial to its final direction, is the equation of the plane relative orbit; and the other equation of condition ($I^2.$), connecting the same two constants with the same extreme distances and with the time, gives the law of the velocity of mutual approach or recess.

We may remark that the part V_i of the whole characteristic function V , which represents the relative action and determines the relative motion in the system, namely,

$$V_i = \frac{m_1 m_2}{m_1 + m_2} \left(h \S + \int_{r_0}^r \varepsilon \, dr \right), \quad (V^2.)$$

may be put, by ($I^2.$), under the form

$$V_i = \frac{m_1 m_2}{m_1 + m_2} \int_{r_0}^r \left(\varepsilon - h \frac{\delta \varepsilon}{\delta h} \right) dr, \quad (W^2.)$$

or finally, by (79.),

$$V_i = 2 \int_{r_0}^r \frac{m_1 m_2 f(r) + H_i}{\varepsilon} dr; \quad (X^2.)$$

the condition ($I^2.$) may also itself be transformed, by (79.), as follows:

$$\S = h \int_{r_0}^r \frac{dr}{r^2 \varepsilon}; \quad (Y^2.)$$

results which all admit of easy verifications. The partial differential equations connected with the law of relative living force, which the characteristic function V_i of relative motion must satisfy, may be put under the following forms :

$$\left. \begin{aligned} \left(\frac{\delta V_i}{\delta r} \right)^2 + \frac{1}{r^2} \left(\frac{\delta V_i}{\delta \S} \right)^2 &= \frac{2}{m_1 + m_2} (U + H_i), \\ \left(\frac{\delta V_i}{\delta r_0} \right)^2 + \frac{1}{r_0^2} \left(\frac{\delta V_i}{\delta \S} \right)^2 &= \frac{2}{m_1 + m_2} (U_0 + H_i); \end{aligned} \right\} \quad (Z^2.)$$

and if the first of the equations of this pair have its variation taken with respect to r and \S , attention being paid to the dynamical meanings of the coefficients of the characteristic function, it will conduct (as in former instances) to the known differential equations of motion of the second order.

On the undisturbed Motion of a Planet or Comet about the Sun: Dependence of the Characteristic Function of such Motion, on the chord and the sum of the Radii.

15. To particularize still further, let

$$f(r) = \frac{1}{r}, \quad (101.)$$

that is, let us consider a binary system, such as a planet or comet and the sun, with the Newtonian law of attraction; and let us put, for abridgement,

$$m_1 + m_2 = \mu, \quad \frac{h^2}{\mu} = p, \quad \frac{-m_1 m_2}{2 H_i} = a. \quad (102.)$$

The characteristic function V_i of relative motion may now be expressed as follows :

$$V_i = \frac{m_1 m_2}{\sqrt{\mu}} \left(\mathfrak{H} \sqrt{p} + \int_{r_0}^r \pm \sqrt{\frac{2}{r} - \frac{1}{a} - \frac{p}{r^2}} \cdot d r \right); \quad \dots \dots \dots \quad (A^3.)$$

in which p is to be considered as a function of the extreme radii vectores r, r_0 , and of their included angle \mathfrak{H} , involving also the quantity a , or the connected quantity H_i , and determined by the condition

$$\mathfrak{H} = \int_{r_0}^r \frac{\pm dr}{\sqrt{\frac{2}{r} - \frac{1}{a} - \frac{p}{r^2}}}, \quad \dots \dots \dots \quad (B^3.)$$

that is, by the derivative of the formula (A³), taken with respect to p : the upper sign being taken in each expression when the distance r is increasing, and the lower sign when that distance is diminishing, and the quantity p being treated as constant in calculating the two definite integrals. It results from the foregoing remarks, that this quantity p is constant also in the sense of being independent of the time, so as not to vary in the course of the motion; and that the condition (B³), connecting this constant with $r r_0 \mathfrak{H} a$, is the equation of the plane relative orbit; which is therefore (as it has long been known to be) an ellipse, hyperbola, or parabola, according as the constant a is positive, negative, or zero, the origin of r being always a focus of the curve, and p being the semiparameter. It results also, that the time of motion may be thus expressed :

$$t = \frac{\delta V_i}{\delta H_i} = \frac{2 a^2}{m_1 m_2} \frac{\delta V_i}{\delta a}, \quad \dots \dots \dots \quad (C^3.)$$

and therefore thus :

$$t = \int_{r_0}^r \frac{\pm dr}{\sqrt{\frac{2}{r} \frac{\mu}{a} - \frac{\mu}{a} - \frac{\mu p}{r^2}}}; \quad \dots \dots \dots \quad (D^3.)$$

which latter is a known expression. Confining ourselves at present to the case $a > 0$, and introducing the known auxiliary quantities called excentricity and excentric anomaly, namely,

$$e = \sqrt{1 - \frac{p}{a}}, \quad \dots \dots \dots \quad (103.)$$

and

$$\nu = \cos^{-1} \left(\frac{a - r}{a e} \right), \quad \dots \dots \dots \quad (104.)$$

which give

$$\pm \sqrt{2 a r - r^2 - p a} = a e \sin \nu, \quad \dots \dots \dots \quad (105.)$$

ν being considered as continually increasing with the time; and therefore, as is well known,

$$r = a (1 - e \cos \nu), \quad r_0 = a (1 - e \cos \nu_0),$$

$$\mathfrak{H} = 2 \tan^{-1} \left\{ \sqrt{\frac{1+e}{1-e}} \tan \frac{\nu}{2} \right\} - 2 \tan^{-1} \left\{ \sqrt{\frac{1+e}{1-e}} \tan \frac{\nu_0}{2} \right\}, \quad \dots \quad (106.)$$

and

$$t = \sqrt{\frac{a^3}{\mu}} \cdot (v - v_0 - e \sin v + e \sin v_0); \quad \dots \dots \dots \dots \quad (107.)$$

we find that this expression for the characteristic function of relative motion,

$$V_i = \frac{m_1 m_2}{\sqrt{\mu}} \int_{r_0}^r \frac{\pm \left(\frac{a}{r} - \frac{1}{a} \right) dr}{\sqrt{\frac{a}{r} - \frac{1}{a} - \frac{p}{r^2}}}, \quad \dots \dots \dots \quad (E^3.)$$

deduced from (A³.) and (B³.), may be transformed as follows :

$$V_i = m_1 m_2 \sqrt{\frac{a}{\mu}} (v - v_0 + e \sin v - e \sin v_0); \quad \dots \dots \quad (F^3.)$$

in which the excentricity e , and the final and initial excentric anomalies v, v_0 , are to be considered as functions of the final and initial radii r, r_0 , and of the included angle \mathfrak{A} , determined by the equations (106.). The expression (F³.) may be thus written :

$$V_i = 2 m_1 m_2 \sqrt{\frac{a}{\mu}} (v_i + e_i \sin v_i), \quad \dots \dots \dots \quad (G^3.)$$

if we put, for abridgement,

$$v_i = \frac{v - v_0}{2}, \quad e_i = e \cos \frac{v + v_0}{2}; \quad \dots \dots \dots \quad (108.)$$

for the complete determination of the characteristic function of the present relative motion, it remains therefore to determine the two variables v_i and e_i , as functions of r, r_0, \mathfrak{A} , or of some other set of quantities which mark the shape and size of the plane triangle bounded by the final and initial elliptic radii vectores and by the elliptic chord.

For this purpose it is convenient to introduce this elliptic chord itself, which we shall call $\pm \pi$, so that

$$r^2 = r^2 + r_0^2 - 2 r r_0 \cos \mathfrak{A}; \quad \dots \dots \dots \quad (109.)$$

because this chord may be expressed as a function of the two variables v_i, e_i , (involving also the mean distance a), as follows. The value (106.) for the angle \mathfrak{A} , that is, by (95.), for $\theta - \theta_0$, gives

$$\theta - 2 \tan^{-1} \left\{ \sqrt{\frac{1+e}{1-e}} \tan \frac{v}{2} \right\} = \theta_0 - 2 \tan^{-1} \left\{ \sqrt{\frac{1+e}{1-e}} \tan \frac{v_0}{2} \right\} = \pi, \quad (110.)$$

π being a new constant independent of the time, namely, one of the values of the polar angle θ , which correspond to the minimum of radius vector; and therefore, by (106.),

$$\left. \begin{aligned} r \cos(\theta - \pi) &= a(\cos v - e), & r \sin(\theta - \pi) &= a \sqrt{1-e^2} \sin v, \\ r_0 \cos(\theta_0 - \pi) &= a(\cos v_0 - e), & r_0 \sin(\theta_0 - \pi) &= a \sqrt{1-e^2} \sin v_0; \end{aligned} \right\} \quad (111.)$$

expressions which give the following value for the square of the elliptic chord :

$$\left. \begin{aligned} \tau^2 &= \{r \cos(\theta - \omega) - r_0 \cos(\theta_0 - \omega)\}^2 + \{r \sin(\theta - \omega) - r_0 \sin(\theta_0 - \omega)\}^2 \\ &= a^2 \{(\cos v - \cos v_0)^2 + (1 - e^2) (\sin v - \sin v_0)^2\} \\ &= 4 a^2 \sin v_i^2 \left\{ \left(\sin \frac{v+v_0}{2} \right)^2 + (1 - e^2) \left(\cos \frac{v+v_0}{2} \right)^2 \right\} \\ &= 4 a^2 (1 - e_i^2) \sin v_i^2; \end{aligned} \right\} \quad (112.)$$

we may also consider τ as having the same sign with $\sin v_i$, if we consider it as alternately positive and negative, in the successive elliptic periods or revolutions, beginning with the initial position.

Besides, if we denote by σ the sum of the two elliptic radii vectores, final and initial, so that

$$\sigma = r + r_0, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (113.)$$

we shall have, with our present abridgements,

$$\sigma = 2 a (1 - e_i \cos v_i); \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (114.)$$

the variables v_i, e_i are therefore functions of σ, τ, a , and consequently the characteristic function V_i is itself a function of those three quantities. We may therefore put

$$V_i = \frac{m_1 m_2 w}{m_1 + m_2}, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (H^3.)$$

w being a function of σ, τ, a , of which the form is to be determined by eliminating v_i, e_i between the three equations,

$$\left. \begin{aligned} w &= 2 \sqrt{\mu a} (v_i + e_i \sin v_i), \\ \sigma &= 2 a (1 - e_i \cos v_i), \\ \tau &= 2 a (1 - e_i^2)^{\frac{1}{2}} \sin v_i; \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (I^3.)$$

and we may consider this new function w as itself a characteristic function of elliptic motion; the law of its variation being expressed as follows, in the notation of the present essay :

$$\delta w = \xi' \delta \xi - \alpha' \delta \alpha + \eta' \delta \eta - \beta' \delta \beta + \zeta' \delta \zeta - \gamma' \delta \gamma + \frac{t \mu \delta a}{2 a^2}. \quad \dots \quad \dots \quad \dots \quad (K^3.)$$

In this expression, $\xi \eta \zeta$ are the relative coordinates of the point m_1 , at the time t , referred to the other attracting point m_2 as an origin, and to any three rectangular axes; $\xi' \eta' \zeta'$ are their rates of increase, or the three rectangular components of final relative velocity; $\alpha \beta \gamma \alpha' \beta' \gamma'$ are the initial values, or values at the time zero, of these relative coordinates and components of relative velocity; a is a quantity independent of the time, namely, the mean distance of the two points m_1, m_2 ; and μ is the sum of their masses. And all the properties of the undisturbed elliptic motion of a planet or comet about the sun may be deduced in a new way, from the simplified characteristic function w , by comparing its variation (K³) with the following other form,

$$\delta w = \frac{\delta w}{\delta \sigma} \delta \sigma + \frac{\delta w}{\delta \tau} \delta \tau + \frac{\delta w}{\delta a} \delta a; \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (L^3.)$$

in which we are to observe that

$$\left. \begin{aligned} \sigma &= \sqrt{\xi^2 + \eta^2 + \zeta^2} + \sqrt{\alpha^2 + \beta^2 + \gamma^2}, \\ \tau &= \pm \sqrt{(\xi - \alpha)^2 + (\eta - \beta)^2 + (\zeta - \gamma)^2}. \end{aligned} \right\} \quad \dots \quad \text{(M}^3\text{.)}$$

By this comparison we are brought back to the general integral equations of the relative motion of a binary system, (89.) and (90.); but we have now the following particular values for the coefficients A, B, C:

$$A = \frac{1}{r} \frac{\delta w}{\delta \sigma} + \frac{1}{\tau} \frac{\delta w}{\delta \tau}, \quad B = \frac{1}{\tau} \frac{\delta w}{\delta \tau}, \quad C = \frac{1}{r_0} \frac{\delta w}{\delta \sigma} + \frac{1}{\tau} \frac{\delta w}{\delta \tau}; \quad \dots \quad \text{(N}^3\text{.)}$$

and with respect to the three partial differential coefficients, $\frac{\delta w}{\delta \sigma}$, $\frac{\delta w}{\delta \tau}$, $\frac{\delta w}{\delta a}$, we have the following relation between them :

$$a \frac{\delta w}{\delta a} + \sigma \frac{\delta w}{\delta \sigma} + \tau \frac{\delta w}{\delta \tau} = \frac{w}{2}, \quad \dots \quad \text{(O}^3\text{.)}$$

the function w being homogeneous of the dimension $\frac{1}{2}$ with respect to the three quantities a , σ , τ ; we have also, by (13.),

$$\frac{\delta w}{\delta \sigma} = \sqrt{\frac{\mu}{a}} \cdot \frac{\sin v_i}{e_i - \cos v_i}, \quad \frac{\delta w}{\delta \tau} = \sqrt{\frac{\mu}{a}} \cdot \frac{\sqrt{1 - e_i^2}}{\cos v_i - e_i}, \quad \dots \quad \text{(P}^3\text{.)}$$

and therefore

$$\frac{\delta w}{\delta \sigma} \frac{\delta w}{\delta \tau} = \frac{-2\mu\tau}{\sigma^2 - \tau^2}, \quad \left(\frac{\delta w}{\delta \sigma} \right)^2 + \left(\frac{\delta w}{\delta \tau} \right)^2 + \frac{\mu}{a} = \frac{4\mu\sigma}{\sigma^2 - \tau^2}, \quad \dots \quad \text{(Q}^3\text{.)}$$

from which may be deduced the following remarkable expressions :

$$\left. \begin{aligned} \left(\frac{\delta w}{\delta \sigma} + \frac{\delta w}{\delta \tau} \right)^2 &= \frac{4\mu}{\sigma + \tau} - \frac{\mu}{a}, \\ \left(\frac{\delta w}{\delta \tau} - \frac{\delta w}{\delta \sigma} \right)^2 &= \frac{4\mu}{\sigma - \tau} - \frac{\mu}{a}. \end{aligned} \right\} \quad \dots \quad \text{(R}^3\text{.)}$$

These expressions will be found to be important in the application of the present method to the theory of elliptic motion.

16. We shall not enter, on this occasion, into any details of such application; but we may remark, that the circumstance of the characteristic function involving only the elliptic chord and the sum of the extreme radii, (besides the mean distance and the sum of the masses,) affords, by our general method, a new proof of the well-known theorem that the elliptic time also depends on the same chord and sum of radii; and gives a new expression for the law of this dependence, namely,

$$t = \frac{2a^2}{\mu} \frac{\delta w}{\delta a}. \quad \dots \quad \text{(S}^3\text{.)}$$

We may remark also, that the same form of the characteristic function of elliptic motion, conducts, by our general method, to the following curious, but not novel property, of the ellipse, that if any two tangents be drawn to such a curve, from any common point outside, these tangents subtend equal angles at one focus;

they subtend also equal angles at the other. Reciprocally, if any plane curve possesses this property, when referred to a fixed point in its own plane, which may be taken as the origin of polar coordinates r, θ , the curve must satisfy the following equation in mixed differences:

$$\cotan\left(\frac{\Delta\theta}{2}\right) \cdot \Delta \frac{1}{r} = (\Delta + 2) \frac{d}{d\theta} \frac{1}{r}, \quad \dots \quad (115.)$$

which may be brought to the following form,

$$\left(\frac{d}{d\theta} + \frac{d^3}{d\theta^3}\right) \frac{1}{r} = 0, \quad \dots \quad (116.)$$

and therefore gives, by integration,

the curve is, consequently, a conic section, and the fixed point is one of its foci.

The properties of parabolic are included as limiting cases in those of elliptic motion, and may be deduced from them by making

and therefore the characteristic function w and the time t , in parabolic as well as in elliptic motion, are functions of the chord and of the sum of the radii. By thus making a infinite in the foregoing expressions, we find, for parabolic motion, the partial differential equations

$$\left(\frac{\delta w}{\delta \sigma} + \frac{\delta w}{\delta \tau}\right)^2 = \frac{4 \mu}{\sigma + \tau}, \quad \left(\frac{\delta w}{\delta \sigma} - \frac{\delta w}{\delta \tau}\right)^2 = \frac{4 \mu}{\sigma - \tau}; \quad \dots \quad (T^3.)$$

and in fact the parabolic form of the simplified characteristic function w may easily be shown to be

$$w = 2\sqrt{\mu}(\sqrt{\sigma+\tau} \mp \sqrt{\sigma-\tau}), \quad \dots \quad (U^3)$$

τ being, as before, the chord, and σ the sum of the radii; while the analogous limit of the expression $(\$^3)$, for the time, is

$$t = \frac{1}{6\sqrt{\mu}} \left\{ (\sigma + \tau)^{\frac{2}{3}} \mp (\sigma - \tau)^{\frac{2}{3}} \right\} : \dots \dots \dots \dots \dots \quad (V3.)$$

which latter is a known expression.

The formulæ (K³.) and (L³.), to the comparison of which we have reduced the study of elliptic motion, extend to hyperbolic motion also; and in any binary system, with Newton's law of attraction, the simplified characteristic function w may be expressed by the definite integral

$$w = \int^{\tau} \sqrt{\frac{\mu}{\varepsilon + \varepsilon_0} - \frac{\mu}{A_0}} \cdot d\tau, \quad \dots \dots \dots \dots \dots \dots \dots \dots \quad (W^3.)$$

this function w being still connected with the relative action V , by the equation (H^3) ; while the time t , which may always be deduced from this function, by the law of varying action, is represented by this other connected integral,

$$t = \frac{1}{4} \int_{-\tau_i}^{\tau} \left(\frac{\mu}{\sigma + \tau} - \frac{\mu}{4a} \right)^{-\frac{1}{2}} d\tau \dots \dots \dots \quad (X^3.)$$

provided that, within the extent of these integrations, the radical does not vanish nor become infinite. When this condition is not satisfied, we may still express the simplified characteristic function w , and the time t , by the following analogous integrals :

$$w = \int_{\tau_i}^{\tau} \pm \sqrt{\frac{2\mu}{\sigma_i} - \frac{\mu}{a}} d\sigma_p \dots \dots \dots \quad (Y^3.)$$

and

$$t = \int_{\tau_i}^{\tau} \pm \left(\frac{2\mu}{\sigma_i} - \frac{\mu}{a} \right)^{-\frac{1}{2}} d\sigma_p \dots \dots \dots \quad (Z^3.)$$

in which we have put for abridgement

$$\sigma_i = \frac{\sigma + \tau}{2}, \quad \tau_i = \frac{\sigma - \tau}{2}, \dots \dots \dots \quad (119.)$$

and in which it is easy to determine the signs of the radicals. But to treat fully of these various transformations would carry us too far at present, for it is time to consider the properties of systems with more points than two.

On Systems of three Points, in general; and on their Characteristic Functions.

17. For any system of three points, the known differential equations of motion of the 2nd order are included in the following formula :

$$\left. \begin{aligned} m_1(x''_1 \delta x_1 + y''_1 \delta y_1 + z''_1 \delta z_1) + m_2(x''_2 \delta x_2 + y''_2 \delta y_2 + z''_2 \delta z_2) \\ + m_3(x''_3 \delta x_3 + y''_3 \delta y_3 + z''_3 \delta z_3) = \delta U, \end{aligned} \right\} \quad (120.)$$

the known force-function U having the form

$$U = m_1 m_2 f^{(1, 2)} + m_1 m_3 f^{(1, 3)} + m_2 m_3 f^{(2, 3)}, \dots \dots \quad (121.)$$

in which $f^{(1, 2)}$, $f^{(1, 3)}$, $f^{(2, 3)}$, are functions respectively of the three following mutual distances of the points of the system :

$$\left. \begin{aligned} r^{(1, 2)} &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}, \\ r^{(1, 3)} &= \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2}, \\ r^{(2, 3)} &= \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2}, \end{aligned} \right\} \quad (122.)$$

the known differential equations of motion are therefore, separately, for the point m_1 ,

$$\left. \begin{aligned} x''_1 &= m_2 \frac{\delta f^{(1, 2)}}{\delta x_1} + m_3 \frac{\delta f^{(1, 3)}}{\delta x_1}, \\ y''_1 &= m_2 \frac{\delta f^{(1, 2)}}{\delta y_1} + m_3 \frac{\delta f^{(1, 3)}}{\delta y_1}, \\ z''_1 &= m_2 \frac{\delta f^{(1, 2)}}{\delta z_1} + m_3 \frac{\delta f^{(1, 3)}}{\delta z_1}, \end{aligned} \right\} \quad \dots \dots \dots \quad (123.)$$

with six other analogous equations for the points m_2 and m_3 ; x''_1 , &c., denoting the

component accelerations of the three points $m_1 m_2 m_3$, or the second differential coefficients of their coordinates, taken with respect to the time. To integrate these equations is to assign, by their means, nine relations between the time t , the three masses $m_1 m_2 m_3$, the nine varying coordinates $x_1 y_1 z_1 x_2 y_2 z_2 x_3 y_3 z_3$, and their nine initial values and nine initial rates of increase, which may be thus denoted, $a_1 b_1 c_1 a_2 b_2 c_2 a_3 b_3 c_3 a'_1 b'_1 c'_1 a'_2 b'_2 c'_2 a'_3 b'_3 c'_3$. The known intermediate integral containing the law of living force, namely,

$$\left. \begin{aligned} & \frac{1}{2} m_1 (x'_1{}^2 + y'_1{}^2 + z'_1{}^2) + \frac{1}{2} m_2 (x'_2{}^2 + y'_2{}^2 + z'_2{}^2) + \frac{1}{2} m_3 (x'_3{}^2 + y'_3{}^2 + z'_3{}^2) \\ & = m_1 m_2 f^{(1, 2)} + m_1 m_3 f^{(1, 3)} + m_2 m_3 f^{(2, 3)} + H, \end{aligned} \right\} \quad (124.)$$

gives the following initial relation :

$$\left. \begin{aligned} & \frac{1}{2} m_1 (a'_1{}^2 + b'_1{}^2 + c'_1{}^2) + \frac{1}{2} m_2 (a'_2{}^2 + b'_2{}^2 + c'_2{}^2) + \frac{1}{2} m_3 (a'_3{}^2 + b'_3{}^2 + c'_3{}^2) \\ & = m_1 m_2 f_0^{(1, 2)} + m_1 m_3 f_0^{(1, 3)} + m_2 m_3 f_0^{(2, 3)} + H, \end{aligned} \right\} \quad (125.)$$

in which $f_0^{(1, 2)}$, $f_0^{(1, 3)}$, $f_0^{(2, 3)}$, are composed of the initial coordinates, in the same manner as $f^{(1, 2)}$, $f^{(1, 3)}$, $f^{(2, 3)}$ are composed of the final coordinates. If then we knew the nine final integrals of the equations of motion of this ternary system, and combined them with the initial form (125.) of the law of living force, we should have ten relations to determine the ten quantities $t a'_1 b'_1 c'_1 a'_2 b'_2 c'_2 a'_3 b'_3 c'_3$, namely, the time and the nine initial components of the velocities of the three points, as functions of the nine final and nine initial coordinates, and of the quantity H , involving also the masses; we could therefore determine whatever else depends on the manner and time of motion of the system, from its initial to its final position, as a function of the same extreme coordinates, and of H . In particular, we could determine the action V , or the accumulated living force of the system, namely,

$$\left. \begin{aligned} V &= m_1 \int_0^t (x'_1{}^2 + y'_1{}^2 + z'_1{}^2) dt + m_2 \int_0^t (x'_2{}^2 + y'_2{}^2 + z'_2{}^2) dt \\ &+ m_3 \int_0^t (x'_3{}^2 + y'_3{}^2 + z'_3{}^2) dt, \end{aligned} \right\} . \quad (A^4.)$$

as a function of these nineteen quantities, $x_1 y_1 z_1 x_2 y_2 z_2 x_3 y_3 z_3 a_1 b_1 c_1 a_2 b_2 c_2 a_3 b_3 c_3 H$; and might then calculate the variation of this function,

$$\left. \begin{aligned} \delta V &= \frac{\delta V}{\delta x_1} \delta x_1 + \frac{\delta V}{\delta y_1} \delta y_1 + \frac{\delta V}{\delta z_1} \delta z_1 + \frac{\delta V}{\delta a_1} \delta a_1 + \frac{\delta V}{\delta b_1} \delta b_1 + \frac{\delta V}{\delta c_1} \delta c_1 \\ &+ \frac{\delta V}{\delta x_2} \delta x_2 + \frac{\delta V}{\delta y_2} \delta y_2 + \frac{\delta V}{\delta z_2} \delta z_2 + \frac{\delta V}{\delta a_2} \delta a_2 + \frac{\delta V}{\delta b_2} \delta b_2 + \frac{\delta V}{\delta c_2} \delta c_2 \\ &+ \frac{\delta V}{\delta x_3} \delta x_3 + \frac{\delta V}{\delta y_3} \delta y_3 + \frac{\delta V}{\delta z_3} \delta z_3 + \frac{\delta V}{\delta a_3} \delta a_3 + \frac{\delta V}{\delta b_3} \delta b_3 + \frac{\delta V}{\delta c_3} \delta c_3 \\ &+ \frac{\delta V}{\delta H} \delta H. \end{aligned} \right\} . \quad (B^4.)$$

But the law of varying action gives, *previously*, the following expression for this variation :

$$\begin{aligned} \delta V = & m_1 (x'_1 \delta x_1 - d'_1 \delta a_1 + y'_1 \delta y_1 - b'_1 \delta b_1 + z'_1 \delta z_1 - c'_1 \delta c_1) \\ & + m_2 (x'_2 \delta x_2 - d'_2 \delta a_2 + y'_2 \delta y_2 - b'_2 \delta b_2 + z'_2 \delta z_2 - c'_2 \delta c_2) \\ & + m_3 (x'_3 \delta x_3 - d'_3 \delta a_3 + y'_3 \delta y_3 - b'_3 \delta b_3 + z'_3 \delta z_3 - c'_3 \delta c_3) \\ & + t \delta H; \end{aligned} \quad (C^4.)$$

and shows, therefore, that the research of all the intermediate and all the final integral equations, of motion of the system, may be reduced, reciprocally, to the search and differentiation of this one characteristic function V ; because if we knew this one function, we should have the nine intermediate integrals of the known differential equations, under the forms

$$\left. \begin{aligned} \frac{\delta V}{\delta x_1} &= m_1 x'_1, \quad \frac{\delta V}{\delta y_1} = m_1 y'_1, \quad \frac{\delta V}{\delta z_1} = m_1 z'_1, \\ \frac{\delta V}{\delta x_2} &= m_2 x'_2, \quad \frac{\delta V}{\delta y_2} = m_2 y'_2, \quad \frac{\delta V}{\delta z_2} = m_2 z'_2, \\ \frac{\delta V}{\delta x_3} &= m_3 x'_3, \quad \frac{\delta V}{\delta y_3} = m_3 y'_3, \quad \frac{\delta V}{\delta z_3} = m_3 z'_3, \end{aligned} \right\} \quad \dots \dots \dots \dots \dots \quad (D^4.)$$

and the nine final integrals under the forms

$$\left. \begin{aligned} \frac{\delta V}{\delta a_1} &= -m_1 d'_1, \quad \frac{\delta V}{\delta b_1} = -m_1 b'_1, \quad \frac{\delta V}{\delta c_1} = -m_1 c'_1, \\ \frac{\delta V}{\delta a_2} &= -m_2 d'_2, \quad \frac{\delta V}{\delta b_2} = -m_2 b'_2, \quad \frac{\delta V}{\delta c_2} = -m_2 c'_2, \\ \frac{\delta V}{\delta a_3} &= -m_3 d'_3, \quad \frac{\delta V}{\delta b_3} = -m_3 b'_3, \quad \frac{\delta V}{\delta c_3} = -m_3 c'_3, \end{aligned} \right\} \quad \dots \dots \dots \dots \quad (E^4.)$$

the auxiliary constant H being to be eliminated, and the time t introduced, by this other equation, which has often occurred in this essay,

$$t = \frac{\delta V}{\delta H}. \quad \dots \dots \dots \dots \quad (E.)$$

The same law of varying action suggests also a method of investigating the form of this characteristic function V , not requiring the previous integration of the known equations of motion; namely, the integration of a pair of partial differential equations connected with the law of living force; which are,

$$\begin{aligned} & \frac{1}{2m_1} \left\{ \left(\frac{\delta V}{\delta x_1} \right)^2 + \left(\frac{\delta V}{\delta y_1} \right)^2 + \left(\frac{\delta V}{\delta z_1} \right)^2 \right\} + \frac{1}{2m_2} \left\{ \left(\frac{\delta V}{\delta x_2} \right)^2 + \left(\frac{\delta V}{\delta y_2} \right)^2 + \left(\frac{\delta V}{\delta z_2} \right)^2 \right\} \\ & + \frac{1}{2m_3} \left\{ \left(\frac{\delta V}{\delta x_3} \right)^2 + \left(\frac{\delta V}{\delta y_3} \right)^2 + \left(\frac{\delta V}{\delta z_3} \right)^2 \right\} = m_1 m_2 f^{(1, 2)} + m_1 m_3 f^{(1, 3)} + m_2 m_3 f^{(2, 3)} + H, \end{aligned} \quad (F^4.)$$

and

$$\begin{aligned} & \frac{1}{2m_1} \left\{ \left(\frac{\delta V}{\delta a_1} \right)^2 + \left(\frac{\delta V}{\delta b_1} \right)^2 + \left(\frac{\delta V}{\delta c_1} \right)^2 \right\} + \frac{1}{2m_2} \left\{ \left(\frac{\delta V}{\delta a_2} \right)^2 + \left(\frac{\delta V}{\delta b_2} \right)^2 + \left(\frac{\delta V}{\delta c_2} \right)^2 \right\} \\ & + \frac{1}{2m_3} \left\{ \left(\frac{\delta V}{\delta a_3} \right)^2 + \left(\frac{\delta V}{\delta b_3} \right)^2 + \left(\frac{\delta V}{\delta c_3} \right)^2 \right\} = m_1 m_2 f_0^{(1, 2)} + m_1 m_3 f_0^{(1, 3)} + m_2 m_3 f_0^{(2, 3)} + H. \end{aligned} \quad (G^4.)$$

And to diminish the difficulty of thus determining the function V , which depends on 18 coordinates, we may separate it, by principles already explained, into a part V_u depending only on the motion of the centre of gravity of the system, and determined by the formula (H¹), and another part V_p depending only on the relative motions of the points of the system about this internal centre, and equal to the accumulated living force, connected with this relative motion only. In this manner the difficulty is reduced to determining the relative action V_i ; and if we introduce the relative co-ordinates

$$\begin{aligned} \xi_1 &= x_1 - x_3, & \eta_1 &= y_1 - y_3, & \zeta_1 &= z_1 - z_3, \\ \xi_2 &= x_2 - x_3, & \eta_2 &= y_2 - y_3, & \zeta_2 &= z_2 - z_3, \end{aligned} \quad \dots \quad (126.)$$

and

$$\begin{aligned} \alpha_1 &= a_1 - a_3, & \beta_1 &= b_1 - b_3, & \gamma_1 &= c_1 - c_3, \\ \alpha_2 &= a_2 - a_3, & \beta_2 &= b_2 - b_3, & \gamma_2 &= c_2 - c_3, \end{aligned} \quad \dots \quad (127.)$$

we easily find, by the principles of the tenth and following numbers, that the function V , may be considered as depending only on these relative coordinates, and on a quantity H , analogous to H (besides the masses of the system); and that it must satisfy two partial differential equations, analogous to (F⁴) and (G⁴), namely,

$$\begin{aligned} & \frac{1}{2m_1} \left\{ \left(\frac{\partial V}{\partial \xi_1} \right)^2 + \left(\frac{\partial V}{\partial \eta_1} \right)^2 + \left(\frac{\partial V}{\partial \zeta_1} \right)^2 \right\} + \frac{1}{2m_2} \left\{ \left(\frac{\partial V}{\partial \xi_2} \right)^2 + \left(\frac{\partial V}{\partial \eta_2} \right)^2 + \left(\frac{\partial V}{\partial \zeta_2} \right)^2 \right\} \\ & + \frac{1}{2m_3} \left\{ \left(\frac{\partial V}{\partial \xi_1} + \frac{\partial V}{\partial \xi_2} \right)^2 + \left(\frac{\partial V}{\partial \eta_1} + \frac{\partial V}{\partial \eta_2} \right)^2 + \left(\frac{\partial V}{\partial \zeta_1} + \frac{\partial V}{\partial \zeta_2} \right)^2 \right\} \\ & = m_1 m_2 f^{(1,2)} + m_1 m_3 f^{(1,3)} + m_2 m_3 f^{(2,3)} + H_i; \end{aligned} \quad \left. \begin{aligned} & \frac{1}{2m_1} \left\{ \left(\frac{\partial V}{\partial \alpha_1} \right)^2 + \left(\frac{\partial V}{\partial \beta_1} \right)^2 + \left(\frac{\partial V}{\partial \gamma_1} \right)^2 \right\} + \frac{1}{2m_2} \left\{ \left(\frac{\partial V}{\partial \alpha_2} \right)^2 + \left(\frac{\partial V}{\partial \beta_2} \right)^2 + \left(\frac{\partial V}{\partial \gamma_2} \right)^2 \right\} \\ & + \frac{1}{2m_3} \left\{ \left(\frac{\partial V}{\partial \alpha_1} + \frac{\partial V}{\partial \alpha_2} \right)^2 + \left(\frac{\partial V}{\partial \beta_1} + \frac{\partial V}{\partial \beta_2} \right)^2 + \left(\frac{\partial V}{\partial \gamma_1} + \frac{\partial V}{\partial \gamma_2} \right)^2 \right\} \\ & = m_1 m_2 f_0^{(1,2)} + m_1 m_3 f_0^{(1,3)} + m_2 m_3 f_0^{(2,3)} + H_i; \end{aligned} \right\} \quad (H^4.) \quad (I^4.)$$

and

$$\begin{aligned} & \frac{1}{2m_1} \left\{ \left(\frac{\partial V}{\partial \alpha_1} \right)^2 + \left(\frac{\partial V}{\partial \beta_1} \right)^2 + \left(\frac{\partial V}{\partial \gamma_1} \right)^2 \right\} + \frac{1}{2m_2} \left\{ \left(\frac{\partial V}{\partial \alpha_2} \right)^2 + \left(\frac{\partial V}{\partial \beta_2} \right)^2 + \left(\frac{\partial V}{\partial \gamma_2} \right)^2 \right\} \\ & + \frac{1}{2m_3} \left\{ \left(\frac{\partial V}{\partial \alpha_1} + \frac{\partial V}{\partial \alpha_2} \right)^2 + \left(\frac{\partial V}{\partial \beta_1} + \frac{\partial V}{\partial \beta_2} \right)^2 + \left(\frac{\partial V}{\partial \gamma_1} + \frac{\partial V}{\partial \gamma_2} \right)^2 \right\} \\ & = m_1 m_2 f_0^{(1,2)} + m_1 m_3 f_0^{(1,3)} + m_2 m_3 f_0^{(2,3)} + H_i; \end{aligned} \quad \left. \begin{aligned} & \frac{1}{2m_1} \left\{ \left(\frac{\partial V}{\partial \xi_1} \right)^2 + \left(\frac{\partial V}{\partial \eta_1} \right)^2 + \left(\frac{\partial V}{\partial \zeta_1} \right)^2 \right\} + \frac{1}{2m_2} \left\{ \left(\frac{\partial V}{\partial \xi_2} \right)^2 + \left(\frac{\partial V}{\partial \eta_2} \right)^2 + \left(\frac{\partial V}{\partial \zeta_2} \right)^2 \right\} \\ & + \frac{1}{2m_3} \left\{ \left(\frac{\partial V}{\partial \xi_1} + \frac{\partial V}{\partial \xi_2} \right)^2 + \left(\frac{\partial V}{\partial \eta_1} + \frac{\partial V}{\partial \eta_2} \right)^2 + \left(\frac{\partial V}{\partial \zeta_1} + \frac{\partial V}{\partial \zeta_2} \right)^2 \right\} \\ & = m_1 m_2 f^{(1,2)} + m_1 m_3 f^{(1,3)} + m_2 m_3 f^{(2,3)} + H_i; \end{aligned} \right\} \quad (K^4.)$$

the law of the variation of this function being, by (Z¹),

$$\begin{aligned} \delta V_i &= t \delta H_i + m_1 (\xi'_1 \delta \xi_1 - \alpha'_1 \delta \alpha_1 + \eta'_1 \delta \eta_1 - \beta'_1 \delta \beta_1 + \zeta'_1 \delta \zeta_1 - \gamma'_1 \delta \gamma_1) \\ & \quad + m_2 (\xi'_2 \delta \xi_2 - \alpha'_2 \delta \alpha_2 + \eta'_2 \delta \eta_2 - \beta'_2 \delta \beta_2 + \zeta'_2 \delta \zeta_2 - \gamma'_2 \delta \gamma_2). \\ & - \frac{1}{m_1 + m_2 + m_3} \left\{ \begin{aligned} & (m_1 \xi'_1 + m_2 \xi'_2) (m_1 \delta \xi_1 + m_2 \delta \xi_2) - (m_1 \alpha'_1 + m_2 \alpha'_2) (m_1 \delta \alpha_1 + m_2 \delta \alpha_2) \\ & + (m_1 \eta'_1 + m_2 \eta'_2) (m_1 \delta \eta_1 + m_2 \delta \eta_2) - (m_1 \beta'_1 + m_2 \beta'_2) (m_1 \delta \beta_1 + m_2 \delta \beta_2) \\ & + (m_1 \zeta'_1 + m_2 \zeta'_2) (m_1 \delta \zeta_1 + m_2 \delta \zeta_2) - (m_1 \gamma'_1 + m_2 \gamma'_2) (m_1 \delta \gamma_1 + m_2 \delta \gamma_2) \end{aligned} \right\} \quad (K^4.) \end{aligned}$$

which resolves itself in the same manner as before into the six intermediate and six final integrals of relative motion, namely, into the following equations :

$$\left. \begin{aligned} \frac{1}{m_1} \frac{\delta V_i}{\delta \xi_1} &= \zeta'_1 - \frac{m_1 \zeta'_1 + m_2 \zeta'_2}{m_1 + m_2 + m_3}; & \frac{1}{m_2} \frac{\delta V_i}{\delta \xi_2} &= \zeta'_2 - \frac{m_1 \zeta'_1 + m_2 \zeta'_2}{m_1 + m_2 + m_3}; \\ \frac{1}{m_1} \frac{\delta V_i}{\delta \eta_1} &= \eta'_1 - \frac{m_1 \eta'_1 + m_2 \eta'_2}{m_1 + m_2 + m_3}; & \frac{1}{m_2} \frac{\delta V_i}{\delta \eta_2} &= \eta'_2 - \frac{m_1 \eta'_1 + m_2 \eta'_2}{m_1 + m_2 + m_3}; \\ \frac{1}{m_1} \frac{\delta V_i}{\delta \zeta'_1} &= \zeta'_1 - \frac{m_1 \zeta'_1 + m_2 \zeta'_2}{m_1 + m_2 + m_3}; & \frac{1}{m_2} \frac{\delta V_i}{\delta \zeta'_2} &= \zeta'_2 - \frac{m_1 \zeta'_1 + m_2 \zeta'_2}{m_1 + m_2 + m_3}; \end{aligned} \right\} \quad \dots \quad (L^*)$$

and

$$\left. \begin{aligned} \frac{-1}{m_1} \frac{\delta V_i}{\delta \alpha'_1} &= \alpha'_1 - \frac{m_1 \alpha'_1 + m_2 \alpha'_2}{m_1 + m_2 + m_3}; & \frac{-1}{m_2} \frac{\delta V_i}{\delta \alpha'_2} &= \alpha'_2 - \frac{m_1 \alpha'_1 + m_2 \alpha'_2}{m_1 + m_2 + m_3}; \\ \frac{-1}{m_1} \frac{\delta V_i}{\delta \beta'_1} &= \beta'_1 - \frac{m_1 \beta'_1 + m_2 \beta'_2}{m_1 + m_2 + m_3}; & \frac{-1}{m_2} \frac{\delta V_i}{\delta \beta'_2} &= \beta'_2 - \frac{m_1 \beta'_1 + m_2 \beta'_2}{m_1 + m_2 + m_3}; \\ \frac{-1}{m_1} \frac{\delta V_i}{\delta \gamma'_1} &= \gamma'_1 - \frac{m_1 \gamma'_1 + m_2 \gamma'_2}{m_1 + m_2 + m_3}; & \frac{-1}{m_2} \frac{\delta V_i}{\delta \gamma'_2} &= \gamma'_2 - \frac{m_1 \gamma'_1 + m_2 \gamma'_2}{m_1 + m_2 + m_3}; \end{aligned} \right\} \quad \dots \quad (M^*)$$

which must be combined with our old formula,

$$\frac{\delta V_i}{\delta H_i} = t. \quad \dots \quad (O^1.)$$

18. The quantity H_i in V_i , and the analogous quantity H_u in V_u , are indeed independent of the time, and do not vary in the course of the motion; but it is required by the spirit of our method, that in deducing the absolute action or original characteristic function V from the two parts V_i and V_u , we should consider these two parts H and H_u of the original quantity H , as functions involving each the nine initial and nine final coordinates of the points of the ternary system; the forms of these two functions, of the eighteen coordinates and of H , being determined by the two conditions,

$$\frac{\delta V_i}{\delta H_i} = \frac{\delta V_u}{\delta H_u}, \quad H_i + H_u = H. \quad \dots \quad (N^4.)$$

However, it results from these conditions, that in taking the variation of the whole original function V , of the first order, with respect to the eighteen coordinates, we may treat the two auxiliary quantities H_i and H_u as constant; and therefore that we have the following expressions for the partial differential coefficients of the first order of V , taken with respect to the coordinates parallel to x ,

$$\left. \begin{aligned} \frac{\delta V}{\delta x_1} &= \frac{\delta V_i}{\delta \xi_1} + \frac{m_1}{m_1 + m_2 + m_3} \frac{\delta V_u}{\delta x_1}, & \frac{\delta V}{\delta \alpha'_1} &= \frac{\delta V_i}{\delta \alpha'_1} + \frac{m_1}{m_1 + m_2 + m_3} \frac{\delta V_u}{\delta \alpha'_1}, \\ \frac{\delta V}{\delta x_2} &= \frac{\delta V_i}{\delta \xi_2} + \frac{m_2}{m_1 + m_2 + m_3} \frac{\delta V_u}{\delta x_2}, & \frac{\delta V}{\delta \alpha'_2} &= \frac{\delta V_i}{\delta \alpha'_2} + \frac{m_2}{m_1 + m_2 + m_3} \frac{\delta V_u}{\delta \alpha'_2}, \\ \frac{\delta V}{\delta x_3} &= -\frac{\delta V_i}{\delta \xi_1} - \frac{\delta V_i}{\delta \xi_2} + \frac{m_3}{m_1 + m_2 + m_3} \frac{\delta V_u}{\delta x_3}, & \frac{\delta V}{\delta \alpha'_3} &= -\frac{\delta V_i}{\delta \alpha'_1} - \frac{\delta V_i}{\delta \alpha'_2} + \frac{m_3}{m_1 + m_2 + m_3} \frac{\delta V_u}{\delta \alpha'_3}, \end{aligned} \right\} \quad (O^4.)$$

together with analogous expressions for the partial differential coefficients of the same order, taken with respect to the other coordinates. Substituting these expressions in the equations of the form (O.), namely, in the following,

$$\left. \begin{aligned} \frac{\delta V}{\delta x_1} + \frac{\delta V}{\delta x_2} + \frac{\delta V}{\delta x_3} + \frac{\delta V}{\delta a_1} + \frac{\delta V}{\delta a_2} + \frac{\delta V}{\delta a_3} &= 0, \\ \frac{\delta V}{\delta y_1} + \frac{\delta V}{\delta y_2} + \frac{\delta V}{\delta y_3} + \frac{\delta V}{\delta b_1} + \frac{\delta V}{\delta b_2} + \frac{\delta V}{\delta b_3} &= 0, \\ \frac{\delta V}{\delta z_1} + \frac{\delta V}{\delta z_2} + \frac{\delta V}{\delta z_3} + \frac{\delta V}{\delta c_1} + \frac{\delta V}{\delta c_2} + \frac{\delta V}{\delta c_3} &= 0, \end{aligned} \right\} \dots \quad (P^4)$$

we find that these equations become identical, because

$$\frac{\delta V_u}{\delta x_u} + \frac{\delta V_u}{\delta a_u} = 0, \quad \frac{\delta V_u}{\delta y_u} + \frac{\delta V_u}{\delta b_u} = 0, \quad \frac{\delta V_u}{\delta z_u} + \frac{\delta V_u}{\delta c_u} = 0. \quad \dots \quad (Q^4)$$

But substituting, in like manner, the expressions (O⁴) in the equations of the form (P₁), of which the first is, for a ternary system,

$$\left. \begin{aligned} x_1 \frac{\delta V}{\delta y_1} - y_1 \frac{\delta V}{\delta x_1} + x_2 \frac{\delta V}{\delta y_2} - y_2 \frac{\delta V}{\delta x_2} + x_3 \frac{\delta V}{\delta y_3} - y_3 \frac{\delta V}{\delta x_3} \\ + a_1 \frac{\delta V}{\delta b_1} - b_1 \frac{\delta V}{\delta a_1} + a_2 \frac{\delta V}{\delta b_2} - b_2 \frac{\delta V}{\delta a_2} + a_3 \frac{\delta V}{\delta b_3} - b_3 \frac{\delta V}{\delta a_3} \end{aligned} \right\} \dots \quad (R^4)$$

and observing that we have

$$x_u \frac{\delta V_u}{\delta y_u} - y_u \frac{\delta V_u}{\delta x_u} + a_u \frac{\delta V_u}{\delta b_u} - b_u \frac{\delta V_u}{\delta a_u} = 0, \quad \dots \quad (S^4)$$

along with two other analogous conditions, we find that the part V_p , or the characteristic function of relative motion of the ternary system, must satisfy the three following conditions, involving its partial differential coefficients of the first order and in the first degree,

$$\left. \begin{aligned} 0 &= \xi_1 \frac{\delta V_I}{\delta \eta_1} - \eta_1 \frac{\delta V_I}{\delta \xi_1} + \xi_2 \frac{\delta V_I}{\delta \eta_2} - \eta_2 \frac{\delta V_I}{\delta \xi_2} + \xi_3 \frac{\delta V_I}{\delta \eta_3} - \eta_3 \frac{\delta V_I}{\delta \xi_3} + \alpha_1 \frac{\delta V_I}{\delta \beta_1} - \beta_1 \frac{\delta V_I}{\delta \alpha_1} + \alpha_2 \frac{\delta V_I}{\delta \beta_2} - \beta_2 \frac{\delta V_I}{\delta \alpha_2}, \\ 0 &= \eta_1 \frac{\delta V_I}{\delta \xi_1} - \xi_1 \frac{\delta V_I}{\delta \eta_1} + \eta_2 \frac{\delta V_I}{\delta \xi_2} - \xi_2 \frac{\delta V_I}{\delta \eta_2} + \eta_3 \frac{\delta V_I}{\delta \xi_3} - \xi_3 \frac{\delta V_I}{\delta \eta_3} + \beta_1 \frac{\delta V_I}{\delta \gamma_1} - \gamma_1 \frac{\delta V_I}{\delta \beta_1} + \beta_2 \frac{\delta V_I}{\delta \gamma_2} - \gamma_2 \frac{\delta V_I}{\delta \beta_2}, \\ 0 &= \xi_1 \frac{\delta V_I}{\delta \xi_1} - \xi_1 \frac{\delta V_I}{\delta \xi_1} + \xi_2 \frac{\delta V_I}{\delta \xi_2} - \xi_2 \frac{\delta V_I}{\delta \xi_2} + \xi_3 \frac{\delta V_I}{\delta \xi_3} - \xi_3 \frac{\delta V_I}{\delta \xi_3} + \alpha_1 \frac{\delta V_I}{\delta \gamma_1} - \gamma_1 \frac{\delta V_I}{\delta \alpha_1} + \alpha_2 \frac{\delta V_I}{\delta \gamma_2} - \gamma_2 \frac{\delta V_I}{\delta \alpha_2}, \end{aligned} \right\} \quad (T^4)$$

which show that this function can depend only on the shape and size of a pentagon, not generally plane, formed by the point m_3 considered as fixed, and by the initial and final positions of the other two points m_1 and m_2 ; for example, the pentagon, of which the corners are, in order, m_3 (m_1) (m_2) m_2 m_1 ; (m_1) and (m_2) denoting the initial positions of the points m_1 and m_2 , referred to m_3 as a fixed origin. The shape and size of this pentagon may be determined by the ten mutual distances of its five points, that is, by the five sides and five diagonals, which may be thus denoted :

$$\left. \begin{aligned} m_3(m_1) &= \sqrt{s_1}, \quad (m_1)(m_2) = \sqrt{s_2}, \quad (m_2)m_2 = \sqrt{s_3}, \quad m_2m_1 = \sqrt{s_4}, \quad m_1m_3 = \sqrt{s_5}, \\ m_3(m_2) &= \sqrt{d_1}, \quad (m_1)m_2 = \sqrt{d_2}, \quad (m_2)m_1 = \sqrt{d_3}, \quad m_2m_3 = \sqrt{d_4}, \quad m_1(m_1) = \sqrt{d_5}, \end{aligned} \right\} \quad (128.)$$

the values of $s_1 \dots d_5$ as functions of the twelve relative coordinates being

$$\left. \begin{aligned} s_1 &= \alpha_1^2 + \beta_1^2 + \gamma_1^2, & s_2 &= (\alpha_2 - \alpha_1)^2 + (\beta_2 - \beta_1)^2 + (\gamma_2 - \gamma_1)^2, \\ s_3 &= (\xi_2 - \alpha_2)^2 + (\eta_2 - \beta_2)^2 + (\zeta_2 - \gamma_2)^2, \\ s_5 &= \xi_1^2 + \eta_1^2 + \zeta_1^2, & s_4 &= (\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2 + (\zeta_1 - \zeta_2)^2, \\ d_1 &= \alpha_2^2 + \beta_2^2 + \gamma_2^2, & d_2 &= (\xi_2 - \alpha_1)^2 + (\eta_2 - \beta_1)^2 + (\zeta_2 - \gamma_1)^2, \\ & & d_3 &= (\xi_1 - \alpha_2)^2 + (\eta_1 - \beta_2)^2 + (\zeta_1 - \gamma_2)^2, \\ d_4 &= \xi_2^2 + \eta_2^2 + \zeta_2^2, & d_5 &= (\xi_1 - \alpha_1)^2 + (\eta_1 - \beta_1)^2 + (\zeta_1 - \gamma_1)^2. \end{aligned} \right\} \quad (129.)$$

These ten distances $\sqrt{s_1}$, &c., are not, however, all independent, but are connected by one equation of condition, namely,

$$\left. \begin{aligned} 0 &= s_1^2 s_3^2 + s_2^2 s_4^2 + s_3^2 s_5^2 + s_4^2 s_1^2 + s_5^2 s_2^2 \\ &+ s_1^2 d_3^2 + s_2^2 d_4^2 + s_3^2 d_5^2 + s_4^2 d_1^2 + s_5^2 d_2^2 \\ &+ d_1^2 d_2^2 + d_2^2 d_3^2 + d_3^2 d_4^2 + d_4^2 d_5^2 + d_5^2 d_1^2 \\ &- 2 s_1^2 s_3 s_4 - 2 s_2^2 s_4 s_5 - 2 s_3^2 s_5 s_1 - 2 s_4^2 s_1 s_2 - 2 s_5^2 s_2 s_3 \\ &- 2 s_1^2 s_3 d_3 - 2 s_2^2 s_4 d_4 - 2 s_3^2 s_5 d_5 - 2 s_4^2 s_1 d_1 - 2 s_5^2 s_2 d_2 \\ &- 2 s_1^2 s_4 d_3 - 2 s_2^2 s_5 d_4 - 2 s_3^2 s_1 d_5 - 2 s_4^2 s_2 d_1 - 2 s_5^2 s_3 d_2 \\ &- 2 s_1 d_2 d_3^2 - 2 s_2 d_3 d_4^2 - 2 s_3 d_4 d_5^2 - 2 s_4 d_5 d_1^2 - 2 s_5 d_1 d_2^2 \\ &- 2 s_1 d_3^2 d_4 - 2 s_2 d_4^2 d_5 - 2 s_3 d_5^2 d_1 - 2 s_4 d_1^2 d_2 - 2 s_5 d_2^2 d_3 \\ &- 2 d_1 d_2^2 d_3 - 2 d_2 d_3^2 d_4 - 2 d_3 d_4^2 d_5 - 2 d_4 d_5^2 d_1 - 2 d_5 d_1^2 d_2 \\ &- 4 s_1 s_3 s_4 d_3 - 4 s_2 s_4 s_5 d_4 - 4 s_3 s_5 s_1 d_5 - 4 s_4 s_1 s_2 d_1 - 4 s_5 s_2 s_3 d_2 \\ &- 4 s_1 d_2 d_3 d_4 - 4 s_2 d_3 d_4 d_5 - 4 s_3 d_4 d_5 d_1 - 4 s_4 d_5 d_1 d_2 - 4 s_5 d_1 d_2 d_3 \\ &- 2 s_1 s_2 s_3 d_4 - 2 s_2 s_3 s_4 d_5 - 2 s_3 s_4 s_5 d_1 - 2 s_4 s_5 s_1 d_2 - 2 s_5 s_1 s_2 d_3 \\ &- 2 s_1 s_3 d_1 d_2 - 2 s_2 s_4 d_2 d_3 - 2 s_3 s_5 d_3 d_4 - 2 s_4 s_1 d_4 d_5 - 2 s_5 s_2 d_5 d_1 \\ &- 2 s_1 d_1 d_3 d_5 - 2 s_2 d_2 d_4 d_1 - 2 s_3 d_3 d_5 d_2 - 2 s_4 d_4 d_1 d_3 - 2 s_5 d_5 d_2 d_4 \\ &+ 2 s_1 s_2 s_3 s_4 + 2 s_2 s_3 s_4 s_5 + 2 s_3 s_4 s_5 s_1 + 2 s_4 s_5 s_1 s_2 + 2 s_5 s_1 s_2 s_3 \\ &+ 2 s_1 s_2 s_4 d_3 + 2 s_2 s_3 s_5 d_4 + 2 s_3 s_4 s_1 d_2 + 2 s_4 s_5 s_2 d_1 + 2 s_5 s_1 s_3 d_2 \\ &+ 2 s_1 s_3 s_4 d_1 + 2 s_2 s_4 s_5 d_2 + 2 s_3 s_5 s_1 d_3 + 2 s_4 s_1 s_2 d_4 + 2 s_5 s_2 s_3 d_5 \\ &+ 2 s_1 s_2 d_3 d_4 + 2 s_2 s_3 d_4 d_5 + 2 s_3 s_4 d_5 d_1 + 2 s_4 s_5 d_1 d_2 + 2 s_5 s_1 s_2 d_3 \\ &+ 2 s_1 s_3 d_2 d_3 + 2 s_2 s_4 d_3 d_4 + 2 s_3 s_5 d_4 d_5 + 2 s_4 s_1 d_5 d_1 + 2 s_5 s_2 d_1 d_2 \\ &+ 2 s_1 s_4 d_1 d_2 + 2 s_2 s_5 d_2 d_3 + 2 s_3 s_1 d_3 d_4 + 2 s_4 s_2 d_4 d_5 + 2 s_5 s_3 d_5 d_1 \\ &+ 2 s_1 s_4 d_1 d_3 + 2 s_2 s_5 d_2 d_4 + 2 s_3 s_1 d_3 d_5 + 2 s_4 s_2 d_4 d_1 + 2 s_5 s_3 d_5 d_2 \\ &+ 2 s_1 s_4 d_2 d_3 + 2 s_2 s_5 d_3 d_4 + 2 s_3 s_1 d_4 d_5 + 2 s_4 s_2 d_5 d_1 + 2 s_5 s_3 d_1 d_2 \\ &+ 2 s_1 s_4 d_3 d_4 + 2 s_2 s_5 d_4 d_5 + 2 s_3 s_1 d_5 d_1 + 2 s_4 s_2 d_1 d_2 + 2 s_5 s_3 d_2 d_3 \\ &+ 2 s_1 d_1 d_2 d_3 + 2 s_2 d_2 d_3 d_4 + 2 s_3 d_3 d_4 d_5 + 2 s_4 d_4 d_5 d_1 + 2 s_5 d_5 d_1 d_2 \\ &+ 2 s_1 d_3 d_4 d_5 + 2 s_2 d_4 d_5 d_1 + 2 s_3 d_5 d_1 d_2 + 2 s_4 d_1 d_2 d_3 + 2 s_5 d_2 d_3 d_4 \\ &+ 2 d_1 d_2 d_3 d_4 + 2 d_2 d_3 d_4 d_5 + 2 d_3 d_4 d_5 d_1 + 2 d_4 d_5 d_1 d_2 + 2 d_5 d_1 d_2 d_3; \end{aligned} \right\} \quad (130.)$$

they may therefore be expressed as functions of nine independent quantities; for example, of four lines and five angles, $r_0^{(1)} r_0^{(2)} r_0^{(2)} \theta_0^{(1)} \theta_0^{(2)} \theta_0^{(2)} \iota$, on which they depend as follows:

$$\left. \begin{aligned} s_1 &= r_0^{(1)2}, \\ s_2 &= r_0^{(1)2} + r_0^{(2)2} - 2 r_0^{(1)} r_0^{(2)} (\cos \theta_0^{(1)} \cos \theta_0^{(2)} + \sin \theta_0^{(1)} \sin \theta_0^{(2)} \cos \iota), \\ s_3 &= r_0^{(2)2} + r_0^{(2)2} - 2 r_0^{(2)} r_0^{(2)} \cos (\theta_0^{(2)} - \theta_0^{(2)}), \\ s_4 &= r_0^{(2)2} + r^{(1)2} - 2 r_0^{(2)} r^{(1)} (\cos \theta_0^{(1)} \cos \theta_0^{(2)} + \sin \theta_0^{(1)} \sin \theta_0^{(2)} \cos \iota), \\ s_5 &= r^{(1)2}, \\ d_1 &= r_0^{(2)2}, \\ d_2 &= r_0^{(2)2} + r_0^{(1)2} - 2 r_0^{(2)} r_0^{(1)} (\cos \theta_0^{(2)} \cos \theta_0^{(1)} + \sin \theta_0^{(2)} \sin \theta_0^{(1)} \cos \iota), \\ d_3 &= r_0^{(2)2} + r^{(1)2} - 2 r_0^{(2)} r^{(1)} (\cos \theta_0^{(2)} \cos \theta_0^{(1)} + \sin \theta_0^{(2)} \sin \theta_0^{(1)} \cos \iota), \\ d_4 &= r^{(2)2}, \\ d_5 &= r^{(1)2} + r_0^{(1)2} - 2 r^{(1)} r_0^{(1)} \cos (\theta_0^{(1)} - \theta_0^{(1)}), \end{aligned} \right\} (131.)$$

the two line-symbols $r^{(1)} r^{(2)}$ denoting, for abridgement, the same two final radii vectores which were before denoted by $r^{(1, 2)} r^{(2, 2)}$, and $r_0^{(1)} r_0^{(2)}$ representing the initial values of these radii; while $\theta^{(1)} \theta^{(2)} \theta_0^{(1)} \theta_0^{(2)}$ are angles made by these four radii, with the line of intersection of the two planes $r_0^{(1)} r^{(1)}, r_0^{(2)} r^{(2)}$; and ι is the inclination of those two planes to each other. We may therefore consider the characteristic function V , of relative motion, for any ternary system, as depending only on these latter lines and angles, along with the quantity H_r .

The reasoning which it has been thought useful to develope here, for any system of three points, attracting or repelling one another according to any functions of their distances, was alluded to, under a more general form, in the twelfth number of this essay; and shows, for example, that the characteristic function of relative motion in a system of four such points, depends on the shape and size of a heptagon, and therefore only on the mutual distances of its seven corners, which are in number $\left(\frac{7 \times 6}{2} =\right) 21$, but are connected by six equations of condition, leaving only fifteen independent. It is easy to extend these remarks to any multiple system.

General method of improving an approximate expression for the Characteristic Function of motion of a System in any problem of Dynamics.

19. The partial differential equation (F.), which the characteristic function V must satisfy, in every dynamical question, may receive some useful general transformations, by the separation of this function V into any two parts

For if we establish, for abridgement, the two following equations of definition,

$$\left. \begin{aligned} T_1 &= \Sigma \cdot \frac{1}{2m} \left(\left(\frac{\delta V_1}{\delta x} \right)^2 + \left(\frac{\delta V_1}{\delta y} \right)^2 + \left(\frac{\delta V_1}{\delta z} \right)^2 \right), \\ T_2 &= \Sigma \cdot \frac{1}{2m} \left(\left(\frac{\delta V_2}{\delta x} \right)^2 + \left(\frac{\delta V_2}{\delta y} \right)^2 + \left(\frac{\delta V_2}{\delta z} \right)^2 \right), \end{aligned} \right\} \dots \dots \dots \quad (V^4)$$

analogous to the relation

$$T = \Sigma \cdot \frac{1}{2m} \left(\left(\frac{\delta V}{\delta x} \right)^2 + \left(\frac{\delta V}{\delta y} \right)^2 + \left(\frac{\delta V}{\delta z} \right)^2 \right), \quad \dots \quad (W^4)$$

which served to transform the law of living force into the partial differential equation (F.) ; we shall have, by (U⁴),

$$T = T_1 + T_2 + \Sigma \cdot \frac{1}{m} \left(\frac{\delta V_1}{\delta x} \frac{\delta V_2}{\delta x} + \frac{\delta V_1}{\delta y} \frac{\delta V_2}{\delta y} + \frac{\delta V_1}{\delta z} \frac{\delta V_2}{\delta z} \right), \dots \quad (X^4),$$

and this expression may be further transformed by the help of the formula (C.), on by the law of varying action. For that law gives the following symbolic equation,

the symbols in both members being prefixed to any one function of the varying co-ordinates of a system, not expressly involving the time; it gives therefore by (U^4) , (V^4) ,

$$\Sigma \cdot \frac{1}{m} \left(\frac{\delta V_1}{\delta x} \frac{\delta V_2}{\delta x} + \frac{\delta V_1}{\delta y} \frac{\delta V_2}{\delta y} + \frac{\delta V_1}{\delta z} \frac{\delta V_2}{\delta z} \right) = \frac{d V_2}{d t} - 2 T_2. \quad \dots \quad (Z^4)$$

In this manner we find the following general and rigorous transformation of the equation (F),

$$\frac{dV_2}{dt} = T - T_1 + T_2; \quad \dots \quad (A^5)$$

T being here retained for the sake of symmetry and conciseness, instead of the equal expression $U + H$. And if we suppose, as we may, that the part V_1 , like the whole function V , is chosen so as to vanish with the time, then the other part V_2 will also have that property, and may be expressed by the definite integral,

$$V_2 = \int_0^t (T - T_1 + T_2) \, dt. \quad \dots \dots \dots \dots \dots \dots \dots \quad (B^5.)$$

More generally, if we employ the principles of the seventh number, and introduce any $3n$ marks $\eta_1, \eta_2, \dots, \eta_{3n}$, of the varying positions of the n points of any system, (whether they be the rectangular coordinates themselves, or any functions of them,) we shall have

and may establish by analogy the two following equations of definition,

$$\left. \begin{array}{l} T_1 = F \left(\frac{\delta V_1}{\delta \eta_1}, \frac{\delta V_1}{\delta \eta_2}, \dots, \frac{\delta V_1}{\delta \eta_{3n}} \right), \\ T_2 = F \left(\frac{\delta V_2}{\delta \eta_1}, \frac{\delta V_2}{\delta \eta_2}, \dots, \frac{\delta V_2}{\delta \eta_{3n}} \right), \end{array} \right\} \dots \dots \dots \dots \dots \dots \quad (D^5).$$

the function F being always rational and integer, and homogeneous of the second dimension; and being therefore such that (besides other properties)

$$T = T_1 + T_2 + \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_1}} \frac{\delta V_2}{\delta \eta_1} + \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_2}} \frac{\delta V_2}{\delta \eta_2} + \dots + \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_{3n}}} \frac{\delta V_2}{\delta \eta_{3n}}, \quad \dots \quad (E5)$$

$$\frac{\delta T}{\delta \frac{\delta V}{\delta \eta_1}} = \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_1}} + \frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_1}}, \dots, \frac{\delta T}{\delta \frac{\delta V}{\delta \eta_{3n}}} = \frac{\delta T_1}{\delta \frac{\delta V_1}{\delta \eta_{3n}}} + \frac{\delta T_2}{\delta \frac{\delta V_2}{\delta \eta_{3n}}}, \dots \dots \dots \quad (F5)$$

and

$$\frac{\frac{\delta T_3}{\delta V_2} \frac{\delta V_2}{\delta \eta_1}}{\frac{\delta V_2}{\delta \eta_1}} + \frac{\frac{\delta T_3}{\delta V_2} \frac{\delta V_3}{\delta \eta_2}}{\frac{\delta V_2}{\delta \eta_2}} + \dots + \frac{\frac{\delta T_3}{\delta V_2} \frac{\delta V_n}{\delta \eta_{3,n}}}{\frac{\delta V_2}{\delta \eta_{3,n}}} = 2 T_2. \quad \dots \quad (G^5)$$

By the principles of the eighth number, we have also,

$$\frac{\delta T}{\delta \frac{\delta V}{\delta \eta_1}} = \eta'_1, \frac{\delta T}{\delta \frac{\delta V}{\delta \eta_2}} = \eta'_2, \dots, \frac{\delta T}{\delta \frac{\delta V}{\delta \eta_{3n}}} = \eta'_{3n}; \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (H^5,$$

and since the meanings of $\eta'_1, \dots, \eta'_{3n}$, give evidently the symbolical equation,

$$\eta'_1 \frac{\delta}{\delta \eta_1} + \eta'_2 \frac{\delta}{\delta \eta_2} + \dots + \eta'_{3n} \frac{\delta}{\delta \eta_{3n}} = \frac{d}{dt}, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (I^5).$$

we see that the equation (A⁵) still holds with the present more general marks of position of a moving system, and gives still the expression (B⁵), supposing only, as before, that the two parts of the whole characteristic function are chosen so as to vanish with the time.

It may not at first sight appear, that this rigorous transformation (B⁵.), of the partial differential equation (F.), or of the analogous equation (T.) with coordinates not rectangular, is likely to assist much in discovering the form of the part V_2 of the characteristic function V , (the other part V_1 being supposed to have been previously assumed;) because it involves under the sign of integration, in the term T_2 , the partial differential coefficients of the sought part V_2 . But if we observe that these unknown coefficients enter only by their squares and products, we shall perceive that it offers a general method of improving an approximation in any problem of dynamics. For if the first part V_1 be an approximate value of the whole sought function V , the second part V_2 will be small, and the term T_2 will not only be also small, but will be in general of a higher order of smallness; we shall therefore in general improve an approximate value V_1 of the characteristic function V , by adding to it the definite integral,

$$V_2 = \int_0^t (T - T_1) dt; \quad \dots \quad (K^5)$$

though this is not, like (B⁵), a perfectly rigorous expression for the remaining part of the function. And in calculating this integral (K⁵), for the improvement of an approximation V_1 , we may employ the following analogous approximations to the rigorous formulae (D.) and (E.),

$$\left. \begin{array}{l} \frac{\delta V_1}{\delta a_1} = -m_1 a'_1; \frac{\delta V_1}{\delta a_2} = -m_2 a'_2; \dots \frac{\delta V_1}{\delta a_n} = -m_n a'_n; \\ \frac{\delta V_1}{\delta b_1} = -m_1 b'_1; \frac{\delta V_1}{\delta b_2} = -m_2 b'_2; \dots \frac{\delta V_1}{\delta b_n} = -m_n b'_n; \\ \frac{\delta V_1}{\delta c_1} = -m_1 c'_1; \frac{\delta V_1}{\delta c_2} = -m_2 c'_2; \dots \frac{\delta V_1}{\delta c_n} = -m_n c'_n; \end{array} \right\} \dots \dots \dots (L^5).$$

and

$$\frac{\delta V_1}{\delta H} = t; \quad \dots \quad (M^5.)$$

or with any other marks of final and initial position, (instead of rectangular coordinates,) the following approximate forms of the rigorous equations (S.),

together with the formula (M⁵); by which new formulæ the manner of motion of the system is approximately though not rigorously expressed.

It is easy to extend these remarks to problems of relative motion, and to show that in such problems we have the rigorous transformation

$$V_{\alpha} = \int_0^t (T_i - T_{\alpha} + T_{\alpha}) dt, \quad \dots \quad (O^5.)$$

and the approximate expression

V_1 being any approximate value of the function V_r of relative motion, and V_2 being the correction of this value; and T_{11}, T_{12} , being homogeneous functions of the second dimension, composed of the partial differential coefficients of these two parts V_{11}, V_{12} in the same way as T_1 is composed of the coefficients of the whole function V_r . These general remarks may usefully be illustrated by a particular but extensive application.

Application of the foregoing method to the case of a Ternary or Multiple System, with any laws of attraction or repulsion, and with one predominant mass.

20. The value (68.), for the relative living force $2T_r$ of a system, reduces itself successively to the following parts, $2T_r^{(1)}, 2T_r^{(2)}, \dots, 2T_r^{(n-1)}$, when we suppose that

all the $n - 1$ first masses vanish, with the exception of each successively; namely, to the part

$$2 T_i^{(1)} = \frac{m_1 m_n}{m_1 + m_n} (\xi_1'^2 + \eta_1'^2 + \zeta_1'^2), \quad \dots \dots \dots \quad (132.)$$

when only m_1, m_n , do not vanish; the part

$$2 T_i^{(2)} = \frac{m_2 m_n}{m_2 + m_n} (\xi_2'^2 + \eta_2'^2 + \zeta_2'^2), \quad \dots \dots \dots \quad (133.)$$

when all but m_2, m_n , vanish; and so on, as far as the part

$$2 T_i^{(n-1)} = \frac{m_{n-1} m_n}{m_{n-1} + m_n} (\xi_{n-1}'^2 + \eta_{n-1}'^2 + \zeta_{n-1}'^2), \quad \dots \dots \quad (134.)$$

which remains, when only the two last masses are retained. The sum of these $n - 1$ parts is not, in general, equal to the whole relative living force $2 T_i$ of the system, with all the n masses retained; but it differs little from that whole when the first $n - 1$ masses are small in comparison with the last mass m_n ; for the rigorous value of this difference is, by (68.), and by (132.) (133.) (134.),

$$\left. \begin{aligned} & 2 T_i - 2 T_i^{(1)} - 2 T_i^{(2)} - \dots - 2 T_i^{(n-1)} = \\ & \frac{2 m_1}{m_n} (T_i^{(1)} - T_i) + \frac{2 m_2}{m_n} (T_i^{(2)} - T_i) + \dots + \frac{2 m_{n-1}}{m_n} (T_i^{(n-1)} - T_i) \\ & + \frac{1}{m_n} \sum_i m_i m_k \{ (\xi_i - \xi_k)^2 + (\eta_i - \eta_k)^2 + (\zeta_i - \zeta_k)^2 \} : \end{aligned} \right\} \quad (135.)$$

an expression which is small of the second order when the $n - 1$ first masses are small of the first order. If, then, we denote by $V_i^{(1)}, V_i^{(2)}, \dots V_i^{(n-1)}$, the relative actions, or accumulated relative living forces, such as they would be in the $n - 1$ binary systems, $(m_1 m_n), (m_2 m_n), \dots (m_{n-1} m_n)$, without the perturbations of the other small masses of the entire multiple system of n points; so that

$$V_i^{(1)} = \int_0^t 2 T_i^{(1)} dt, \quad V_i^{(2)} = \int_0^t 2 T_i^{(2)} dt, \dots V_i^{(n-1)} = \int_0^t 2 T_i^{(n-1)} dt, \quad (Q^5.)$$

the perturbations being neglected in calculating these $n - 1$ definite integrals; we shall have, as an approximate value for the whole relative action V_i of the system, the sum V_{il} of its values for these separate binary systems,

$$V_{il} = V_i^{(1)} + V_i^{(2)} + \dots + V_i^{(n-1)}. \quad \dots \dots \dots \quad (R^5)$$

This sum, by our theory of binary systems, may be otherwise expressed as follows:

$$V_{il} = \frac{m_1 m_n w^{(1)}}{m_1 + m_n} + \frac{m_2 m_n w^{(2)}}{m_2 + m_n} + \dots + \frac{m_{n-1} m_n w^{(n)}}{m_{n-1} + m_n}, \quad \dots \dots \quad (S^5.)$$

if we put for abridgement

$$\left. \begin{aligned} w^{(1)} &= h^{(1)} \mathbb{S}^{(1)} + \int_{r_0^{(1)}}^{r^{(1)}} r'^{(1)} dr^{(1)}, \\ w^{(2)} &= h^{(2)} \mathbb{S}^{(2)} + \int_{r_0^{(2)}}^{r^{(2)}} r'^{(2)} dr^{(2)}, \\ &\dots \\ w^{(n-1)} &= h^{(n-1)} \mathbb{S}^{(n-1)} + \int_{r_0^{(n-1)}}^{r^{(n-1)}} r'^{(n-1)} dr^{(n-1)}. \end{aligned} \right\} \quad (\text{T}^5)$$

In this expression,

$$\left. \begin{aligned} r'^{(1)} &= \pm \sqrt{2(m_1 + m_n) f^{(1)} + 2g^{(1)} - \frac{h^{(1)2}}{r^{(1)2}}}, \\ &\dots \\ r'^{(n-1)} &= \pm \sqrt{2(m_{n-1} + m_n) f^{(n-1)} + 2g^{(n-1)} - \frac{h^{(n-1)2}}{r^{(n-1)2}}}. \end{aligned} \right\} . . . \quad (\text{U}^5)$$

$r^{(1)}, \dots r^{(n-1)}$, being abridged expressions for the distances $r^{(1, n)}, \dots r^{(n-1, n)}$, and $f^{(1)}, \dots f^{(n-1)}$, being abridgements for the functions $f^{(1, n)}, \dots f^{(n-1, n)}$, of these distances, of which the derivatives, according as they are negative or positive, express the laws of attraction or repulsion: we have also introduced $2n - 2$ auxiliary quantities $h^{(1)} g^{(1)} \dots h^{(n-1)} g^{(n-1)}$, to be eliminated or determined by the following equations of condition:

$$\left. \begin{aligned} 0 &= \mathbb{S}^{(1)} + \int_{r_0^{(1)}}^{r^{(1)}} \frac{\delta r'^{(1)}}{\delta h^{(1)}} dr^{(1)}, \\ 0 &= \mathbb{S}^{(2)} + \int_{r_0^{(2)}}^{r^{(2)}} \frac{\delta r'^{(2)}}{\delta h^{(2)}} dr^{(2)}, \\ &\dots \\ 0 &= \mathbb{S}^{(n-1)} + \int_{r_0^{(n-1)}}^{r^{(n-1)}} \frac{\delta r'^{(n-1)}}{\delta h^{(n-1)}} dr^{(n-1)}, \end{aligned} \right\} \quad (\text{V}^5)$$

and

$$\int_{r_0^{(1)}}^{r^{(1)}} \frac{dr^{(1)}}{r'^{(1)}} = \int_{r_0^{(2)}}^{r^{(2)}} \frac{dr^{(2)}}{r'^{(2)}} = \dots = \int_{r_0^{(n-1)}}^{r^{(n-1)}} \frac{dr^{(n-1)}}{r'^{(n-1)}}, \quad \quad (\text{W}^5)$$

or

$$\frac{\delta w^{(1)}}{\delta g^{(1)}} = \frac{\delta w^{(2)}}{\delta g^{(2)}} = \dots = \frac{\delta w^{(n-1)}}{\delta g^{(n-1)}}, \quad \quad (\text{X}^5)$$

along with this last condition,

$$\frac{m_1 g^{(1)}}{m_1 + m_n} + \frac{m_2 g^{(2)}}{m_2 + m_n} + \frac{m_3 g^{(3)}}{m_3 + m_n} + \dots + \frac{m_{n-1} g^{(n-1)}}{m_{n-1} + m_n} = \frac{H_i}{m_n}; \quad \quad (\text{Y}^5)$$

and we have denoted by $\mathbb{S}^{(1)}, \dots \mathbb{S}^{(n-1)}$, the angles which the final distances $r^{(1)}, \dots r^{(n-1)}$, of the first $n - 1$ points from the last or n th point of the system, make

respectively with the initial distances corresponding, namely, $r_0^{(1)}, \dots, r_0^{(n-1)}$. The variation of the sum V_{A} is, by (S⁵),

$$\delta V_{\text{A}} = \frac{m_1 m_n \delta w^{(1)}}{m_1 + m_n} + \frac{m_2 m_n \delta w^{(2)}}{m_2 + m_n} + \dots + \frac{m_{n-1} m_n \delta w^{(n-1)}}{m_{n-1} + m_n}; \quad \dots \quad (\text{Z}^5)$$

in which, by the equations of condition, we may treat all the auxiliary quantities $h^{(1)} g^{(1)} \dots h^{(n-1)} g^{(n-1)}$ as constant, if H_i be considered as given: so that the part of this variation δV_{A} , which depends on the variations of the final relative coordinates, may be put under the form,

$$\left. \begin{aligned} \delta_{\xi_1, \eta_1, \zeta_1} V_{\text{A}} &= \frac{m_1 m_n}{m_1 + m_n} \left(\frac{\delta w^{(1)}}{\delta \xi_1} \delta \xi_1 + \frac{\delta w^{(1)}}{\delta \eta_1} \delta \eta_1 + \frac{\delta w^{(1)}}{\delta \zeta_1} \delta \zeta_1 \right) \\ &+ \frac{m_2 m_n}{m_2 + m_n} \left(\frac{\delta w^{(2)}}{\delta \xi_2} \delta \xi_2 + \frac{\delta w^{(2)}}{\delta \eta_2} \delta \eta_2 + \frac{\delta w^{(2)}}{\delta \zeta_2} \delta \zeta_2 \right) \\ &+ \dots \\ &+ \frac{m_{n-1} m_n}{m_{n-1} + m_n} \left(\frac{\delta w^{(n-1)}}{\delta \xi_{n-1}} \delta \xi_{n-1} + \frac{\delta w^{(n-1)}}{\delta \eta_{n-1}} \delta \eta_{n-1} + \frac{\delta w^{(n-1)}}{\delta \zeta_{n-1}} \delta \zeta_{n-1} \right). \end{aligned} \right\} \quad (\text{A}^6)$$

By the equations (T⁵.) (U⁵.), or by the theory of binary systems, we have, rigorously,

$$\left. \begin{aligned} \left(\frac{\delta w^{(1)}}{\delta \xi_1} \right)^2 + \left(\frac{\delta w^{(1)}}{\delta \eta_1} \right)^2 + \left(\frac{\delta w^{(1)}}{\delta \zeta_1} \right)^2 &= 2(m_1 + m_n) f^{(1)} + 2g^{(1)}; \\ \left(\frac{\delta w^{(2)}}{\delta \xi_2} \right)^2 + \left(\frac{\delta w^{(2)}}{\delta \eta_2} \right)^2 + \left(\frac{\delta w^{(2)}}{\delta \zeta_2} \right)^2 &= 2(m_2 + m_n) f^{(2)} + 2g^{(2)}; \\ \dots \\ \left(\frac{\delta w^{(n-1)}}{\delta \xi_{n-1}} \right)^2 + \left(\frac{\delta w^{(n-1)}}{\delta \eta_{n-1}} \right)^2 + \left(\frac{\delta w^{(n-1)}}{\delta \zeta_{n-1}} \right)^2 &= 2(m_{n-1} + m_n) f^{(n-1)} + 2g^{(n-1)}; \end{aligned} \right\} \quad (\text{B}^6)$$

and the rigorous law of relative living force for the whole multiple system, is

$$T_i = U + H_i, \quad \dots \quad (50.)$$

in which

$$U = m_n(m_1 f^{(1)} + m_2 f^{(2)} + \dots + m_{n-1} f^{(n-1)}) + \sum_i m_i m_k f^{(i, k)}, \quad (\text{C}^6)$$

and

$$\left. \begin{aligned} T_i &= \frac{1}{2} \left(\frac{1}{m_1} + \frac{1}{m_n} \right) \left\{ \left(\frac{\delta V_i}{\delta \xi_1} \right)^2 + \left(\frac{\delta V_i}{\delta \eta_1} \right)^2 + \left(\frac{\delta V_i}{\delta \zeta_1} \right)^2 \right\} \\ &+ \frac{1}{2} \left(\frac{1}{m_2} + \frac{1}{m_n} \right) \left\{ \left(\frac{\delta V_i}{\delta \xi_2} \right)^2 + \left(\frac{\delta V_i}{\delta \eta_2} \right)^2 + \left(\frac{\delta V_i}{\delta \zeta_2} \right)^2 \right\} \\ &+ \dots + \frac{1}{2} \left(\frac{1}{m_{n-1}} + \frac{1}{m_n} \right) \left\{ \left(\frac{\delta V_i}{\delta \xi_{n-1}} \right)^2 + \left(\frac{\delta V_i}{\delta \eta_{n-1}} \right)^2 + \left(\frac{\delta V_i}{\delta \zeta_{n-1}} \right)^2 \right\} \\ &+ \frac{1}{m_n} \sum_i \left(\frac{\delta V_i}{\delta \xi_i} \frac{\delta V_i}{\delta \xi_k} + \frac{\delta V_i}{\delta \eta_i} \frac{\delta V_i}{\delta \eta_k} + \frac{\delta V_i}{\delta \zeta_i} \frac{\delta V_i}{\delta \zeta_k} \right). \end{aligned} \right\} \quad (\text{D}^6)$$

We have therefore, by changing in this last expression the coefficients of the cha-

racteristic function V_i to those of its first part V_{i1} , and by attending to the foregoing equations,

$$T_{i1} = m_n \sum_i m_i f^{(i)} + H_i + m_n \sum_i \frac{m_i}{m_n + m_i} \frac{m_k}{m_n + m_k} \left(\frac{\delta w^{(i)}}{\delta \xi_i} \frac{\delta w^{(k)}}{\delta \xi_k} + \frac{\delta w^{(i)}}{\delta \eta_i} \frac{\delta w^{(k)}}{\delta \eta_k} + \frac{\delta w^{(i)}}{\delta \zeta_i} \frac{\delta w^{(k)}}{\delta \zeta_k} \right); \quad (E^6)$$

and consequently

$$T_i - T_{i1} = \sum_i m_i m_k \left\{ f^{(i,k)} - \frac{m_n}{(m_n + m_i)(m_n + m_k)} \left(\frac{\delta w^{(i)}}{\delta \xi_i} \frac{\delta w^{(k)}}{\delta \xi_k} + \frac{\delta w^{(i)}}{\delta \eta_i} \frac{\delta w^{(k)}}{\delta \eta_k} + \frac{\delta w^{(i)}}{\delta \zeta_i} \frac{\delta w^{(k)}}{\delta \zeta_k} \right) \right\}. \quad (F^6)$$

The general transformation of the foregoing number gives therefore, rigorously, for the remaining part V_{i2} of the characteristic function V_i of relative motion of the multiple system, the equation

$$V_{i2} = \int_0^t T_{i2} dt + \sum_i m_i m_k \int_0^t \left\{ f^{(i,k)} - \frac{\frac{\delta w^{(i)}}{\delta \xi_i} \frac{\delta w^{(k)}}{\delta \xi_k} + \frac{\delta w^{(i)}}{\delta \eta_i} \frac{\delta w^{(k)}}{\delta \eta_k} + \frac{\delta w^{(i)}}{\delta \zeta_i} \frac{\delta w^{(k)}}{\delta \zeta_k}}{\frac{1}{m_n} (m_n + m_i)(m_n + m_k)} \right\} dt; \quad (G^6)$$

and, approximately, the expression

$$V_{i2} = \sum_i m_i m_k \int_0^t \left\{ f^{(i,k)} - \frac{1}{m_n} (\zeta'_i \zeta'_k + \eta'_i \eta'_k + \zeta'_i \zeta'_k) \right\} dt; \quad (H^6)$$

with which last expression we may combine the following approximate formulae belonging in rigour to binary systems only,

$$\zeta'_i = \frac{\delta w^{(i)}}{\delta \xi_i}, \quad \eta'_i = \frac{\delta w^{(i)}}{\delta \eta_i}, \quad \zeta'_i = \frac{\delta w^{(i)}}{\delta \zeta_i}, \quad \dots \quad (I^6)$$

$$\alpha'_i = -\frac{\delta w^{(i)}}{\delta \alpha_i}, \quad \beta'_i = -\frac{\delta w^{(i)}}{\delta \beta_i}, \quad \gamma'_i = -\frac{\delta w^{(i)}}{\delta \gamma_i}, \quad \dots \quad (K^6)$$

and

$$t = \frac{\delta w^{(i)}}{\delta g^{(i)}}. \quad \dots \quad (L^6)$$

We have also, rigorously, for binary systems, the following differential equations of motion of the second order,

$$\zeta''_i = (m_n + m_i) \frac{\delta f^{(i)}}{\delta \xi_i}; \quad \eta''_i = (m_n + m_i) \frac{\delta f^{(i)}}{\delta \eta_i}; \quad \zeta''_i = (m_n + m_i) \frac{\delta f^{(i)}}{\delta \zeta_i}; \quad \dots \quad (M^6)$$

which enable us to transform in various ways the approximate expression (H⁶). Thus, in the case of a ternary system, with any laws of attraction or repulsion, but with one predominant mass m_3 , the *disturbing part* V_{i2} of the characteristic function V_i of relative motion, may be put under the form

$$V_{i2} = m_1 m_2 W, \quad \dots \quad (N^6)$$

in which the coefficient W may approximately be expressed as follows:

$$W = \int_0^t \left\{ f^{(1,2)} - \frac{1}{m_3} (\zeta'_1 \zeta'_2 + \eta'_1 \eta'_2 + \zeta'_1 \zeta'_2) \right\} dt, \quad \dots \quad (O^6)$$

or thus :

$$W = \int_0^t \left(f^{(1,2)} + \xi_2 \frac{\delta f^{(1)}}{\delta \xi_1} + \eta_2 \frac{\delta f^{(1)}}{\delta \eta_1} + \zeta_2 \frac{\delta f^{(1)}}{\delta \zeta_1} \right) dt \\ - \frac{1}{m_3} \left(\xi_2 \frac{\delta w^{(1)}}{\delta \xi_1} + \eta_2 \frac{\delta w^{(1)}}{\delta \eta_1} + \zeta_2 \frac{\delta w^{(1)}}{\delta \zeta_1} + \alpha_2 \frac{\delta w^{(1)}}{\delta \alpha_1} + \beta_2 \frac{\delta w^{(1)}}{\delta \beta_1} + \gamma_2 \frac{\delta w^{(1)}}{\delta \gamma_1} \right), \right\} \quad (P^6)$$

or finally,

$$W = \int_0^t \left(f^{(1,2)} + \xi_1 \frac{\delta f^{(2)}}{\delta \xi_2} + \eta_1 \frac{\delta f^{(2)}}{\delta \eta_2} + \zeta_1 \frac{\delta f^{(2)}}{\delta \zeta_2} \right) dt \\ - \frac{1}{m_3} \left(\xi_1 \frac{\delta w^{(2)}}{\delta \xi_2} + \eta_1 \frac{\delta w^{(2)}}{\delta \eta_2} + \zeta_1 \frac{\delta w^{(2)}}{\delta \zeta_2} + \alpha_1 \frac{\delta w^{(2)}}{\delta \alpha_2} + \beta_1 \frac{\delta w^{(2)}}{\delta \beta_2} + \gamma_1 \frac{\delta w^{(2)}}{\delta \gamma_2} \right). \right\} \quad (Q^6)$$

In general, for a multiple system, we may put

$$V_{i2} = \Sigma_i m_i m_k W^{(i,k)}; \quad \dots \quad (R^6)$$

and approximately,

$$W^{(i,k)} = \int_0^t \left(f^{(i,k)} + \xi_k \frac{\delta f^{(i)}}{\delta \xi_i} + \eta_k \frac{\delta f^{(i)}}{\delta \eta_i} + \zeta_k \frac{\delta f^{(i)}}{\delta \zeta_i} \right) dt \\ - \frac{1}{m_n} \left(\xi_k \frac{\delta w^{(i)}}{\delta \xi_i} + \eta_k \frac{\delta w^{(i)}}{\delta \eta_i} + \zeta_k \frac{\delta w^{(i)}}{\delta \zeta_i} + \alpha_k \frac{\delta w^{(i)}}{\delta \alpha_i} + \beta_k \frac{\delta w^{(i)}}{\delta \beta_i} + \gamma_k \frac{\delta w^{(i)}}{\delta \gamma_i} \right), \right\} \quad (S^6)$$

or

$$W^{(i,k)} = \int_0^t \left(f^{(i,k)} + \xi_i \frac{\delta f^{(k)}}{\delta \xi_k} + \eta_i \frac{\delta f^{(k)}}{\delta \eta_k} + \zeta_i \frac{\delta f^{(k)}}{\delta \zeta_k} \right) dt \\ - \frac{1}{m_n} \left(\xi_i \frac{\delta w^{(k)}}{\delta \xi_k} + \eta_i \frac{\delta w^{(k)}}{\delta \eta_k} + \zeta_i \frac{\delta w^{(k)}}{\delta \zeta_k} + \alpha_i \frac{\delta w^{(k)}}{\delta \alpha_k} + \beta_i \frac{\delta w^{(k)}}{\delta \beta_k} + \gamma_i \frac{\delta w^{(k)}}{\delta \gamma_k} \right). \right\} \quad (T^6)$$

Rigorous transition from the theory of Binary to that of Multiple Systems, by means of the disturbing part of the whole Characteristic Function; and approximate expressions for the perturbations.

21. The three equations (K⁶.) when the auxiliary constant $g^{(i)}$ is eliminated by the formula (L⁶.), are rigorously (by our theory) the three final integrals of the three known equations of the second order (M⁶.), for the relative motion of the binary system ($m_i m_n$); and give, for such a system, the three varying relative coordinates $\xi_i \eta_i \zeta_i$ as functions of their initial values and initial rates of increase $\alpha_i \beta_i \gamma_i \alpha'_i \beta'_i \gamma'_i$, and of the time t . In like manner the three equations (I⁶.), when $g^{(i)}$ is eliminated by (L⁶.), are rigorously the three intermediate integrals of the same known differential equations of motion of the same binary system. These integrals, however, cease to be rigorous when we introduce the perturbations of the relative motion of this partial or binary system ($m_i m_n$), arising from the attractions or repulsions of the other points m_k , of the whole proposed multiple system; but they may be corrected and rendered rigorous by employing the remaining part V_{i2} of the whole characteristic

function of relative motion V_2 , along with the principal part or approximate value V_{21} .

The equations (X¹.) (Y¹.) of the twelfth number, give rigorously

$$\xi'_i = \frac{1}{m_i} \frac{\delta V_i}{\delta \xi_i} + \frac{1}{m_n} \sum_j \frac{\delta V_j}{\delta \xi_i}, \quad \eta'_i = \frac{1}{m_i} \frac{\delta V_i}{\delta \eta_i} + \frac{1}{m_n} \sum_j \frac{\delta V_j}{\delta \eta_i}, \quad \zeta'_i = \frac{1}{m_i} \frac{\delta V_i}{\delta \zeta_i} + \frac{1}{m_n} \sum_j \frac{\delta V_j}{\delta \zeta_i}, \quad . \quad (U^6.)$$

and

$$-\alpha'_i = \frac{1}{m_i} \frac{\delta V_i}{\delta \alpha'_i} + \frac{1}{m_n} \sum_j \frac{\delta V_j}{\delta \alpha'_i}, \quad -\beta'_i = \frac{1}{m_i} \frac{\delta V_i}{\delta \beta'_i} + \frac{1}{m_n} \sum_j \frac{\delta V_j}{\delta \beta'_i}, \quad -\gamma'_i = \frac{1}{m_i} \frac{\delta V_i}{\delta \gamma'_i} + \frac{1}{m_n} \sum_j \frac{\delta V_j}{\delta \gamma'_i}, \quad (V^6)$$

and therefore, by (A⁶),

$$\left. \begin{aligned} \frac{\delta w^{(i)}}{\delta \xi_i} &= \xi'_i - \Sigma_{ii} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \xi_k} - \frac{1}{m_i} \frac{\delta V_{i2}}{\delta \xi_i} - \frac{1}{m_n} \sum_i \frac{\delta V_{i2}}{\delta \xi_i}, \\ \frac{\delta w^{(i)}}{\delta \eta_i} &= \eta'_i - \Sigma_{ii} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \eta_k} - \frac{1}{m_i} \frac{\delta V_{i2}}{\delta \eta_i} - \frac{1}{m_n} \sum_i \frac{\delta V_{i2}}{\delta \eta_i}, \\ \frac{\delta w^{(i)}}{\delta \zeta_i} &= \zeta'_i - \Sigma_{ii} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \zeta_k} - \frac{1}{m_i} \frac{\delta V_{i2}}{\delta \zeta_i} - \frac{1}{m_n} \sum_i \frac{\delta V_{i2}}{\delta \zeta_i}, \end{aligned} \right\} \quad \dots \quad (W^6)$$

and similarly

$$\left. \begin{aligned} -\frac{\delta w^{(i)}}{\delta \alpha_i} &= \omega'_i + \Sigma_u \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \alpha_k} + \frac{1}{m_i} \frac{\delta V_R}{\delta \alpha_i} + \frac{1}{m_n} \Sigma_i \frac{\delta V_R}{\delta \alpha_i}, \\ -\frac{\delta w^{(i)}}{\delta \beta_i} &= \beta'_i + \Sigma_u \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \beta_k} + \frac{1}{m_i} \frac{\delta V_R}{\delta \beta_i} + \frac{1}{m_n} \Sigma_i \frac{\delta V_R}{\delta \beta_i}, \\ -\frac{\delta w^{(i)}}{\delta \gamma_i} &= \gamma'_i + \Sigma_u \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \gamma_k} + \frac{1}{m_i} \frac{\delta V_R}{\delta \gamma_i} + \frac{1}{m_n} \Sigma_i \frac{\delta V_R}{\delta \gamma_i}, \end{aligned} \right\} \quad \dots \quad (N^6)$$

the sign of summation Σ_n referring only to the disturbing masses m_k , to the exclusion of m_i and m_n ; and these equations (W⁶.) (X⁶.) are the rigorous formulæ, corresponding to the approximate relations (I⁶.) (K⁶.) In like manner, the formula (L⁶.) for the time of motion in a binary system, which is only an approximation when the system is considered as multiple, may be rigorously corrected for perturbation by adding to it an analogous term deduced from the disturbing part V_2 of the whole characteristic function; that is, by changing it to the following:

which gives, for this other coefficient of $w^{(i)}$, the corrected and rigorous expression

$$\frac{\delta w^{(i)}}{\delta g^{(i)}} = t - \frac{\delta V_{i2}}{\delta H_i} \dots \dots \dots \dots \dots \dots \dots \quad (Z^6.)$$

V_2 being here supposed so chosen as to be rigorously the correction of V_A . If therefore by the theory of binary systems, or by eliminating $g^{(i)}$ between the four equations (K⁶), (L⁶), we have deduced expressions for the three varying relative coordinates $\xi_i \eta_i \zeta_i$ as functions of the time t , and of the six initial quantities $\alpha_i \beta_i \gamma_i \alpha'_i \beta'_i \gamma'_i$, which may be thus denoted,

$$\left. \begin{aligned} \xi_i &= \varphi_1 (\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i, t), \\ \eta_i &= \varphi_2 (\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i, t), \\ \zeta_i &= \varphi_3 (\alpha_i, \beta_i, \gamma_i, \alpha'_i, \beta'_i, \gamma'_i, t); \end{aligned} \right\} \quad \dots \quad (A^7.)$$

we shall know that the following relations are rigorously and *identically* true,

$$\left. \begin{aligned} \xi_i &= \varphi_1 \left(\alpha_i, \beta_i, \gamma_i, -\frac{\delta w^{(i)}}{\delta \alpha_i}, -\frac{\delta w^{(i)}}{\delta \beta_i}, -\frac{\delta w^{(i)}}{\delta \gamma_i}, \frac{\delta w^{(i)}}{\delta g^{(i)}} \right), \\ \eta_i &= \varphi_2 \left(\alpha_i, \beta_i, \gamma_i, -\frac{\delta w^{(i)}}{\delta \alpha_i}, -\frac{\delta w^{(i)}}{\delta \beta_i}, -\frac{\delta w^{(i)}}{\delta \gamma_i}, \frac{\delta w^{(i)}}{\delta g^{(i)}} \right), \\ \zeta_i &= \varphi_3 \left(\alpha_i, \beta_i, \gamma_i, -\frac{\delta w^{(i)}}{\delta \alpha_i}, -\frac{\delta w^{(i)}}{\delta \beta_i}, -\frac{\delta w^{(i)}}{\delta \gamma_i}, \frac{\delta w^{(i)}}{\delta g^{(i)}} \right), \end{aligned} \right\} \quad \dots \quad (B^7.)$$

and consequently that these relations will still be rigorously true when we substitute for the four coefficients of $w^{(i)}$ their rigorous values ($X^6.$) and ($Z^6.$) for the case of a multiple system. We may thus retain in rigour for any multiple system the final integrals ($A^7.$) of the motion of a binary system, if only we add to the initial components $\alpha'_i, \beta'_i, \gamma'_i$ of relative velocity, and to the time t , the following perturbational terms :

$$\left. \begin{aligned} \Delta \alpha'_i &= \Sigma_u \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \alpha_k} + \frac{1}{m_i} \frac{\delta V_{i2}}{\delta \alpha_i} + \frac{1}{m_n} \Sigma_l \frac{\delta V_{i2}}{\delta \alpha_l}, \\ \Delta \beta'_i &= \Sigma_u \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \beta_k} + \frac{1}{m_i} \frac{\delta V_{i2}}{\delta \beta_i} + \frac{1}{m_n} \Sigma_l \frac{\delta V_{i2}}{\delta \beta_l}, \\ \Delta \gamma'_i &= \Sigma_u \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \gamma_k} + \frac{1}{m_i} \frac{\delta V_{i2}}{\delta \gamma_i} + \frac{1}{m_n} \Sigma_l \frac{\delta V_{i2}}{\delta \gamma_l}, \end{aligned} \right\} \quad \dots \quad (C^7.)$$

and

$$\Delta t = -\frac{\delta V_{i2}}{\delta H_i}. \quad \dots \quad (D^7.)$$

In the same way, if the theory of binary systems, or the elimination of $g^{(i)}$ between the four equations ($I^6.$) ($L^6.$), has given three intermediate integrals, of the form

$$\left. \begin{aligned} \xi'_i &= \psi_1 (\xi_p, \eta_p, \zeta_p, \alpha_p, \beta_p, \gamma_p, t), \\ \eta'_i &= \psi_2 (\xi_p, \eta_p, \zeta_p, \alpha_p, \beta_p, \gamma_p, t), \\ \zeta'_i &= \psi_3 (\xi_p, \eta_p, \zeta_p, \alpha_p, \beta_p, \gamma_p, t), \end{aligned} \right\} \quad \dots \quad (E^7.)$$

we can conclude that the following equations are rigorous and identical,

$$\left. \begin{aligned} \frac{\delta w^{(i)}}{\delta \xi'_i} &= \psi_1 \left(\xi_p, \eta_p, \zeta_p, \alpha_p, \beta_p, \gamma_p, \frac{\delta w^{(i)}}{\delta g^{(i)}} \right), \\ \frac{\delta w^{(i)}}{\delta \eta'_i} &= \psi_2 \left(\xi_p, \eta_p, \zeta_p, \alpha_p, \beta_p, \gamma_p, \frac{\delta w^{(i)}}{\delta g^{(i)}} \right), \\ \frac{\delta w^{(i)}}{\delta \zeta'_i} &= \psi_3 \left(\xi_p, \eta_p, \zeta_p, \alpha_p, \beta_p, \gamma_p, \frac{\delta w^{(i)}}{\delta g^{(i)}} \right), \end{aligned} \right\} \quad \dots \quad (F^7.)$$

and must therefore be still true, when, in passing to a multiple system, we change the coefficients of $w^{(k)}$ to their rigorous values (W^k) (Z^k) . The three intermediate integrals (E^k) of the motion of a binary system may therefore be adapted rigorously to the case of a multiple system, by first adding to the time t the perturbational term (D^k) , and afterwards adding to the resulting values of the final components of relative velocity the terms

$$\left. \begin{aligned} \Delta \xi'_i &= \Sigma_{\mu} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \xi'_k} + \frac{1}{m_i} \frac{\delta V_{\mu 2}}{\delta \xi'_i} + \frac{1}{m_n} \Sigma_{\nu} \frac{\delta V_{\nu 2}}{\delta \xi'_i}, \\ \Delta \eta'_i &= \Sigma_{\mu} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \eta'_k} + \frac{1}{m_i} \frac{\delta V_{\mu 2}}{\delta \eta'_i} + \frac{1}{m_n} \Sigma_{\nu} \frac{\delta V_{\nu 2}}{\delta \eta'_i}, \\ \Delta \zeta'_i &= \Sigma_{\mu} \cdot \frac{m_k}{m_k + m_n} \frac{\delta w^{(k)}}{\delta \zeta'_k} + \frac{1}{m_i} \frac{\delta V_{\mu 2}}{\delta \zeta'_i} + \frac{1}{m_n} \Sigma_{\nu} \frac{\delta V_{\nu 2}}{\delta \zeta'_i}. \end{aligned} \right\} \quad \dots \quad (G^k)$$

22. To derive now, from these rigorous results, some useful approximate expressions, we shall neglect, in the perturbations, the terms which are of the second order, with respect to the small masses of the system, and with respect to the constant $2H$, of relative living force, which is easily seen to be small of the same order as the masses: and then the perturbations of the coordinates, deduced by the method that has been explained, become

$$\left. \begin{aligned} \Delta \xi_i &= \frac{\delta \xi_i}{\delta \alpha'_i} \Delta \alpha'_i + \frac{\delta \xi_i}{\delta \beta'_i} \Delta \beta'_i + \frac{\delta \xi_i}{\delta \gamma'_i} \Delta \gamma'_i + \frac{\delta \xi_i}{\delta t} \Delta t, \\ \Delta \eta_i &= \frac{\delta \eta_i}{\delta \alpha'_i} \Delta \alpha'_i + \frac{\delta \eta_i}{\delta \beta'_i} \Delta \beta'_i + \frac{\delta \eta_i}{\delta \gamma'_i} \Delta \gamma'_i + \frac{\delta \eta_i}{\delta t} \Delta t, \\ \Delta \zeta_i &= \frac{\delta \zeta_i}{\delta \alpha'_i} \Delta \alpha'_i + \frac{\delta \zeta_i}{\delta \beta'_i} \Delta \beta'_i + \frac{\delta \zeta_i}{\delta \gamma'_i} \Delta \gamma'_i + \frac{\delta \zeta_i}{\delta t} \Delta t, \end{aligned} \right\} \quad \dots \quad (H^k)$$

in which we may employ, instead of the rigorous values (C^k) for $\Delta \alpha'_i$, $\Delta \beta'_i$, $\Delta \gamma'_i$, the following approximate values :

$$\left. \begin{aligned} \Delta \alpha'_i &= \Sigma_{\mu} \frac{m_k}{m_n} \frac{\delta w^{(k)}}{\delta \alpha_k} + \frac{1}{m_i} \frac{\delta V_{\mu 2}}{\delta \alpha_i}, \\ \Delta \beta'_i &= \Sigma_{\mu} \frac{m_k}{m_n} \frac{\delta w^{(k)}}{\delta \beta_k} + \frac{1}{m_i} \frac{\delta V_{\mu 2}}{\delta \beta_i}, \\ \Delta \gamma'_i &= \Sigma_{\mu} \frac{m_k}{m_n} \frac{\delta w^{(k)}}{\delta \gamma_k} + \frac{1}{m_i} \frac{\delta V_{\mu 2}}{\delta \gamma_i}. \end{aligned} \right\} \quad \dots \quad (I^k)$$

To calculate the four coefficients

$$\frac{\delta V_{\mu 2}}{\delta \alpha_i}, \frac{\delta V_{\mu 2}}{\delta \beta_i}, \frac{\delta V_{\mu 2}}{\delta \gamma_i}, \frac{\delta V_{\mu 2}}{\delta t},$$

which enter into the values (I^k) (D^k) , we may consider $V_{\mu 2}$ by (R^k) (T^k) , and by the theory of binary systems, as a function of the initial and final relative coordinates, and initial components of relative velocities, involving also expressly the time t , and the

$n - 2$ auxiliary quantities of the form $g^{(k)}$; and then we are to consider those initial components and auxiliary quantities and the time, as depending themselves on the initial and final coordinates, and on H_i . But it is not difficult to prove, by the foregoing principles, that when t and $g^{(k)}$ are thus considered, their variations are, in the present order of approximation,

$$\frac{\delta t}{\delta g^{(k)}} = \frac{\Sigma_i \cdot m \left(\frac{\delta^2 w}{\delta g^{(k)} \delta g^{(k)}} \right)^{-1} \delta_i \frac{\delta w}{\delta g^{(k)}} + \delta H_i}{\Sigma_i \cdot m \left(\frac{\delta^2 w}{\delta g^{(k)} \delta g^{(k)}} \right)^{-1}} \quad \dots \quad \text{(K7.)}$$

and

$$\delta g^{(k)} = \left(\frac{\delta^2 w^{(k)}}{\delta g^{(k)} \delta g^{(k)}} \right)^{-1} \left(\delta t - \delta_i \frac{\delta w^{(k)}}{\delta g^{(k)}} \right), \quad \dots \quad \text{(L7.)}$$

the sign of variation δ_i referring only to the initial and final coordinates; and also that

$$\frac{\delta^2 w^{(i)}}{\delta g^{(i)} \delta t} = \frac{\delta^2 w^{(i)}}{\delta \alpha_i \delta g^{(i)}} \frac{\delta \xi_i}{\delta \alpha_i} + \frac{\delta^2 w^{(i)}}{\delta \beta_i \delta g^{(i)}} \frac{\delta \xi_i}{\delta \beta_i} + \frac{\delta^2 w^{(i)}}{\delta \gamma_i \delta g^{(i)}} \frac{\delta \xi_i}{\delta \gamma_i}, \quad \dots \quad \text{(M7.)}$$

along with two other analogous relations between the coefficients of the two other coordinates $\eta^{(i)}$, $\zeta^{(i)}$; from which it follows that t and $g^{(k)}$, and therefore $\alpha'_i \beta'_i \gamma'_i$, may be treated as constant, in taking the variation of the disturbing part V_2 , for the purpose of calculating the perturbations (H7.); and that the terms involving Δt are destroyed by other terms. We may therefore put simply

$$\left. \begin{aligned} \Delta \xi_i &= \frac{\delta \xi_i}{\delta \alpha'_i} \Delta \alpha'_i + \frac{\delta \xi_i}{\delta \beta'_i} \Delta \beta'_i + \frac{\delta \xi_i}{\delta \gamma'_i} \Delta \gamma'_i, \\ \Delta \eta_i &= \frac{\delta \eta_i}{\delta \alpha'_i} \Delta \alpha'_i + \frac{\delta \eta_i}{\delta \beta'_i} \Delta \beta'_i + \frac{\delta \eta_i}{\delta \gamma'_i} \Delta \gamma'_i, \\ \Delta \zeta_i &= \frac{\delta \zeta_i}{\delta \alpha'_i} \Delta \alpha'_i + \frac{\delta \zeta_i}{\delta \beta'_i} \Delta \beta'_i + \frac{\delta \zeta_i}{\delta \gamma'_i} \Delta \gamma'_i, \end{aligned} \right\} \quad \dots \quad \text{(N7.)}$$

employing for $\Delta \alpha'_i$ the following new expression,

$$\Delta \alpha'_i = \Sigma_u \cdot m_k \left\{ \int_0^t \frac{\delta R^{(i, k)}}{\delta \alpha_i} dt + \frac{\delta \alpha'_i}{\delta \alpha_i} \int_0^t \frac{\delta R^{(i, k)}}{\delta \alpha'_i} dt + \frac{\delta \beta'_i}{\delta \alpha_i} \int_0^t \frac{\delta R^{(i, k)}}{\delta \beta'_i} dt + \frac{\delta \gamma'_i}{\delta \alpha_i} \int_0^t \frac{\delta R^{(i, k)}}{\delta \gamma'_i} dt \right\} \quad \dots \quad \text{(O7.)}$$

together with analogous expressions for $\Delta \beta'_i$, $\Delta \gamma'_i$, in which the sign of summation Σ_u refers to the disturbing masses, and in which the quantity

$$R^{(i, k)} = f^{(i, k)} + \xi_i \frac{\delta f^{(k)}}{\delta \xi_k} + \eta_i \frac{\delta f^{(k)}}{\delta \eta_k} + \zeta_i \frac{\delta f^{(k)}}{\delta \zeta_k} \quad \dots \quad \text{(P7.)}$$

is considered as depending on $\alpha_i \beta_i \gamma_i \alpha'_i \beta'_i \gamma'_i \alpha_k \beta_k \gamma_k \alpha'_k \beta'_k \gamma'_k t$, by the theory of binary systems, while $\alpha'_i \beta'_i \gamma'_i$ are considered as depending, by the same rules, on $\alpha_i \beta_i \gamma_i \xi_i \eta_i \zeta_i$ and t .

It may also be easily shown, that

$$\frac{\delta \xi_i}{\delta \alpha'_i} \frac{\delta \alpha'_i}{\delta \alpha_i} + \frac{\delta \xi_i}{\delta \beta'_i} \frac{\delta \beta'_i}{\delta \beta_i} + \frac{\delta \xi_i}{\delta \gamma'_i} \frac{\delta \gamma'_i}{\delta \gamma_i} = - \frac{\delta \xi_i}{\delta \alpha_i}; \quad \dots \dots \dots \dots \dots \quad (Q^7)$$

with other analogous equations: the perturbation of the coordinate ξ_i may therefore be thus expressed,

$$\left. \begin{aligned} \Delta \xi_i &= \Sigma_{ii} \cdot m_k \left\{ \frac{\delta \xi_i}{\delta \alpha'_i} \int_0^t \frac{\delta R^{(i, k)}}{\delta \alpha_i} dt - \frac{\delta \xi_i}{\delta \alpha_i} \int_0^t \frac{\delta R^{(i, k)}}{\delta \alpha'_i} dt \right. \\ &\quad \left. + \frac{\delta \xi_i}{\delta \beta'_i} \int_0^t \frac{\delta R^{(i, k)}}{\delta \beta_i} dt - \frac{\delta \xi_i}{\delta \beta_i} \int_0^t \frac{\delta R^{(i, k)}}{\delta \beta'_i} dt \right. \\ &\quad \left. + \frac{\delta \xi_i}{\delta \gamma'_i} \int_0^t \frac{\delta R^{(i, k)}}{\delta \gamma_i} dt - \frac{\delta \xi_i}{\delta \gamma_i} \int_0^t \frac{\delta R^{(i, k)}}{\delta \gamma'_i} dt \right\}, \end{aligned} \right\} \quad \dots \dots \dots \quad (R^7)$$

and the perturbations of the two other coordinates may be expressed in an analogous manner.

It results from the same principles, that in taking the first differentials of these perturbations (R^7), the integrals may be treated as constant; and therefore that we may either represent the change of place of the disturbed point m_i in its relative orbit about m_n , by altering a little the initial components of velocity without altering the initial position, and then employing the rules of binary systems; or calculate at once the perturbations of place and of velocity, by employing the same rules, and altering at once the initial position and initial velocity. If we adopt the former of these two methods, we are to employ the expressions (O^7), which may be thus summed up,

$$\left. \begin{aligned} \Delta \alpha'_i &= \Sigma_{ii} \cdot m_k \frac{\delta}{\delta \alpha_i} \int_0^t R^{(i, k)} dt, \\ \Delta \beta'_i &= \Sigma_{ii} \cdot m_k \frac{\delta}{\delta \beta_i} \int_0^t R^{(i, k)} dt, \\ \Delta \gamma'_i &= \Sigma_{ii} \cdot m_k \frac{\delta}{\delta \gamma_i} \int_0^t R^{(i, k)} dt; \end{aligned} \right\} \quad \dots \dots \dots \dots \dots \quad (S^7)$$

and if we adopt the latter method, we are to make,

$$\left. \begin{aligned} \Delta \alpha'_i &= \Sigma_{ii} \cdot m_k \int_0^t \frac{\delta R^{(i, k)}}{\delta \alpha_i} dt, \quad \Delta \alpha_i = - \Sigma_{ii} \cdot m_k \int_0^t \frac{\delta R^{(i, k)}}{\delta \alpha'_i} dt, \\ \Delta \beta'_i &= \Sigma_{ii} \cdot m_k \int_0^t \frac{\delta R^{(i, k)}}{\delta \beta_i} dt, \quad \Delta \beta_i = - \Sigma_{ii} \cdot m_k \int_0^t \frac{\delta R^{(i, k)}}{\delta \beta'_i} dt, \\ \Delta \gamma'_i &= \Sigma_{ii} \cdot m_k \int_0^t \frac{\delta R^{(i, k)}}{\delta \gamma_i} dt, \quad \Delta \gamma_i = - \Sigma_{ii} \cdot m_k \int_0^t \frac{\delta R^{(i, k)}}{\delta \gamma'_i} dt. \end{aligned} \right\} \quad \dots \dots \dots \quad (T^7)$$

The latter was the method of LAGRANGE: the former is suggested more immediately by the principles of the present essay.

General introduction of the Time, into the expression of the Characteristic Function in any dynamical problem.

23. Before we conclude this sketch of our general method in dynamics, it will be proper to notice briefly a transformation of the characteristic function, which may be used in all applications. This transformation consists in putting, generally,

$$V = t H + S, \quad \dots \quad (U^7)$$

and considering the part S , namely, the definite integral

as a function of the initial and final coordinates and of the time, of which the variation is, by our law of varying action,

$$\mathbf{\dot{S}} = -\mathbf{H} dt + \sum_i m (x' \mathbf{\dot{a}}_x - a' \mathbf{\dot{a}}_a + y' \mathbf{\dot{a}}_y - b' \mathbf{\dot{a}}_b + z' \mathbf{\dot{a}}_z - c' \mathbf{\dot{a}}_c). \quad (\text{W}^7)$$

The partial differential coefficients of the first order of this auxiliary function S , are hence,

$$\frac{\delta S}{\delta x_i} = m_i x'_i, \quad \frac{\delta S}{\delta y_i} = m_i y'_i, \quad \frac{\delta S}{\delta z_i} = m_i z'_i; \quad \dots \quad (Y7.)$$

and

$$\frac{\delta S}{\delta a_i} = -m_i a'_i, \quad \frac{\delta S}{\delta b_i} = -m_i b'_i, \quad \frac{\delta S}{\delta c_i} = -m_i c'_i. \quad \dots \quad (Z7.)$$

These last expressions (Z⁷.) are forms for the final integrals of motion of any system, corresponding to the result of elimination of H between the equations (D.) and (E.); and the expressions (Y⁷.) are forms for the intermediate integrals, more convenient in many respects than the forms already employed.

24. The limits of the present essay do not permit us here to develop the consequences of these new expressions. We can only observe, that the auxiliary function S must satisfy the two following equations, in partial differentials of the first order, analogous to, and deduced from, the equations (F.) and (G.):

$$\frac{\delta S}{\delta t} + \Sigma \cdot \frac{1}{\alpha m} \left\{ \left(\frac{\delta S}{\delta x} \right)^2 + \left(\frac{\delta S}{\delta y} \right)^2 + \left(\frac{\delta S}{\delta z} \right)^2 \right\} = U, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (A^8.)$$

and

$$\frac{\delta S}{\delta t} + \Sigma \cdot \frac{1}{2\pi m} \left\{ \left(\frac{\delta S}{\delta a} \right)^2 + \left(\frac{\delta S}{\delta b} \right)^2 + \left(\frac{\delta S}{\delta c} \right)^2 \right\} = U_0; \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (B^8.)$$

and that to correct an approximate value S_1 of S , in the integration of these equations, or to find the remaining part S_2 , if

we may employ the symbolic equation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \Sigma \cdot \frac{1}{m} \left(\frac{\partial S}{\partial x} \frac{\partial}{\partial x} + \frac{\partial S}{\partial y} \frac{\partial}{\partial y} + \frac{\partial S}{\partial z} \frac{\partial}{\partial z} \right); \quad \dots \quad (D^s)$$

which gives, rigorously,

$$\frac{dS_2}{dt} = U - U_1 + \Sigma \cdot \frac{1}{2m} \left\{ \left(\frac{\partial S_2}{\partial x} \right)^2 + \left(\frac{\partial S_2}{\partial y} \right)^2 + \left(\frac{\partial S_2}{\partial z} \right)^2 \right\} \quad \dots \quad (E^s)$$

if we establish by analogy the definition

$$U_1 = \frac{\partial S_1}{\partial t} + \Sigma \cdot \frac{1}{2m} \left\{ \left(\frac{\partial S_1}{\partial x} \right)^2 + \left(\frac{\partial S_1}{\partial y} \right)^2 + \left(\frac{\partial S_1}{\partial z} \right)^2 \right\}; \quad \dots \quad (F^s)$$

and therefore approximately

$$S_2 = \int_0^t (U - U_1) dt, \quad \dots \quad (G^s)$$

the parts $S_1 S_2$ being chosen so as to vanish with the time. These remarks may all be extended easily, so as to embrace relative and polar coordinates, and other marks of position, and offer a new and better way of investigating the orbits and perturbations of a system, by a new and better form of the function and method of this Essay.

March 29, 1834.

VII. *Second Essay on a General Method in Dynamics. By WILLIAM ROWAN HAMILTON, Member of several Scientific Societies in Great Britain and in Foreign Countries, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland. Communicated by Captain BEAUFORT, R.N. F.R.S.*

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Introductory Remarks.

THE former Essay* contained a general method for reducing all the most important problems of dynamics to the study of one characteristic function, one central or radical relation. It was remarked at the close of that Essay, that many eliminations required by this method in its first conception, might be avoided by a general transformation, introducing the time explicitly into a part S of the whole characteristic function V ; and it is now proposed to fix the attention chiefly on this part S , and to call it the *Principal Function*. The properties of this part or function S , which were noticed briefly in the former Essay, are now more fully set forth; and especially its uses in questions of perturbation, in which it dispenses with many laborious and circuitous processes, and enables us to express accurately the disturbed configuration of a system by the rules of undisturbed motion, if only the initial components of velocities be changed in a suitable manner. Another manner of extending rigorously to disturbed motion the rules of undisturbed, by the gradual variation of elements, in number double the number of the coordinates or other marks of position of the system, which was first invented by LAGRANGE, and was afterwards improved by POISSON, is considered in this Second Essay under a form perhaps a little more general; and the general method of calculation which has already been applied to other analogous questions in optics and in dynamics by the author of the present Essay, is now applied to the integration of the equations which determine these elements. This general method is founded chiefly on a combination of the principles of variations with those of partial differentials, and may furnish, when it shall be matured by the labours of other analysts, a separate branch of algebra, which may be called perhaps the *Calculus of Principal Functions*; because, in all the chief applications of algebra to physics, and in a very extensive class of purely mathematical questions, it reduces the determination of many mutually connected functions to the search and study of one principal or central relation. When applied to the integration of the equations of varying elements, it suggests, as is now shown, the consideration

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of a certain *Function of Elements*, which may be variously chosen, and may either be rigorously determined, or at least approached to, with an indefinite accuracy, by a corollary of the general method. And to illustrate all these new general processes, but especially those which are connected with problems of perturbation, they are applied in this Essay to a very simple example, suggested by the motions of projectiles, the parabolic path being treated as the undisturbed. As a more important example, the problem of determining the motions of a ternary or multiple system, with any laws of attraction or repulsion, and with one predominant mass, which was touched upon in the former Essay, is here resumed in a new way, by forming and integrating the differential equations of a new set of varying elements, entirely distinct in theory (though little differing in practice) from the elements conceived by LAGRANGE, and having this advantage, that the differentials of all the new elements for *both* the disturbed and disturbing masses may be expressed by the coefficients of *one* disturbing function.

Transformations of the Differential Equations of Motion of an Attracting or Repelling System.

1. It is well known to mathematicians, that the differential equations of motion of any system of free points, attracting or repelling one another according to any functions of their distances, and not disturbed by any foreign force, may be comprised in the following formula :

$$\Sigma . m (x'' \delta x + y'' \delta y + z'' \delta z) = \delta U: \dots \dots \dots \dots \dots \quad (1.)$$

the sign of summation Σ extending to all the points of the system ; m being, for any one such point, the constant called its mass, and $x y z$ being its rectangular coordinates ; while $x'' y'' z''$ are the accelerations, or second differential coefficients taken with respect to the time, and $\delta x, \delta y, \delta z$ are any arbitrary infinitesimal variations of those coordinates, and U is a certain *force-function*, introduced into dynamics by LAGRANGE, and involving the masses and mutual distances of the several points of the system. If the number of those points be n , the formula (1.) may be decomposed into $3n$ ordinary differential equations of the second order, between the coordinates and the time,

$$m_i x''_i = \frac{\delta U}{\delta x_i}; \quad m_i y''_i = \frac{\delta U}{\delta y_i}; \quad m_i z''_i = \frac{\delta U}{\delta z_i}; \quad \dots \dots \dots \dots \quad (2.)$$

and to integrate these differential equations of motion of an attracting or repelling system, or some transformations of these, is the chief and perhaps ultimately the only problem of mathematical dynamics.

2. To facilitate and generalize the solution of this problem, it is useful to express previously the $3n$ rectangular coordinates $x y z$ as functions of $3n$ other and more general marks of position $\eta_1 \eta_2 \dots \eta_{3n}$; and then the differential equations of motion take this more general form, discovered by LAGRANGE,

$$\frac{d}{dt} \frac{\delta T}{\delta \eta'_i} - \frac{\delta T}{\delta \eta_i} = \frac{\delta U}{\delta \eta_i}, \quad \dots \dots \dots \dots \dots \dots \dots \quad (3.)$$

in which

$$T = \frac{1}{2} \Sigma . m (x'^2 + y'^2 + z'^2). \quad \dots \dots \dots \dots \dots \dots \quad (4.)$$

For, from the equations (2.) or (1.),

$$\left. \begin{aligned} \frac{\delta U}{\delta \eta_i} &= \Sigma . m \left(x'' \frac{\delta x}{\delta \eta_i} + y'' \frac{\delta y}{\delta \eta_i} + z'' \frac{\delta z}{\delta \eta_i} \right) \\ &= \frac{d}{dt} \Sigma . m \left(x' \frac{\delta x}{\delta \eta_i} + y' \frac{\delta y}{\delta \eta_i} + z' \frac{\delta z}{\delta \eta_i} \right) \\ &\quad - \Sigma . m \left(x' \frac{d}{dt} \frac{\delta x}{\delta \eta_i} + y' \frac{d}{dt} \frac{\delta y}{\delta \eta_i} + z' \frac{d}{dt} \frac{\delta z}{\delta \eta_i} \right); \end{aligned} \right\} \quad \dots \dots \dots \quad (5.)$$

in which

$$\left. \begin{aligned} \Sigma . m \left(x' \frac{\delta x}{\delta \eta_i} + y' \frac{\delta y}{\delta \eta_i} + z' \frac{\delta z}{\delta \eta_i} \right) \\ = \Sigma . m \left(x' \frac{\delta x'}{\delta \eta'_i} + y' \frac{\delta y'}{\delta \eta'_i} + z' \frac{\delta z'}{\delta \eta'_i} \right) = \frac{\delta T}{\delta \eta'_i}, \end{aligned} \right\} \quad \dots \dots \dots \quad (6.)$$

and

$$\left. \begin{aligned} \Sigma . m \left(x' \frac{d}{dt} \frac{\delta x}{\delta \eta_i} + y' \frac{d}{dt} \frac{\delta y}{\delta \eta_i} + z' \frac{d}{dt} \frac{\delta z}{\delta \eta_i} \right) \\ = \Sigma . m \left(x' \frac{\delta x'}{\delta \eta_i} + y' \frac{\delta y'}{\delta \eta_i} + z' \frac{\delta z'}{\delta \eta_i} \right) = \frac{\delta T}{\delta \eta_i}, \end{aligned} \right\} \quad \dots \dots \dots \quad (7.)$$

T being here considered as a function of the $6n$ quantities of the forms η' and η , obtained by introducing into its definition (4.), the values

$$x' = \eta'_1 \frac{\delta x}{\delta \eta'_1} + \eta'_2 \frac{\delta x}{\delta \eta'_2} + \dots + \eta'_{3n} \frac{\delta x}{\delta \eta'_{3n}}, \text{ &c.} \quad \dots \dots \dots \quad (8.)$$

A different proof of this important transformation (3.) is given in the Mécanique Analytique.

3. The function T being homogeneous of the second dimension with respect to the quantities η' , must satisfy the condition

$$2T = \Sigma . \eta' \frac{\delta T}{\delta \eta'}; \quad \dots \dots \dots \dots \dots \dots \quad (9.)$$

and since the variation of the same function T may evidently be expressed as follows,

$$\delta T = \Sigma \left(\frac{\delta T}{\delta \eta'} \delta \eta' + \frac{\delta T}{\delta \eta} \delta \eta \right), \quad \dots \dots \dots \dots \dots \dots \quad (10.)$$

we see that this variation may be expressed in this other way,

$$\delta T = \Sigma \left(\eta' \frac{\delta T}{\delta \eta'} - \frac{\delta T}{\delta \eta} \delta \eta \right). \quad \dots \dots \dots \dots \dots \quad (11.)$$

If then we put, for abridgement,

$$\frac{\delta T}{\delta \eta'_1} = \varpi_1, \dots, \frac{\delta T}{\delta \eta'_{3n}} = \varpi_{3n}, \quad \dots \dots \dots \dots \dots \quad (12.)$$

and consider T (as we may) as a function of the following form,

$$T = F(\varpi_1, \varpi_2, \dots, \varpi_{3n}, \eta_1, \eta_2, \dots, \eta_{3n}), \dots \quad (13.)$$

we see that

$$\frac{\delta F}{\delta \varpi_1} = \eta'_1, \dots, \frac{\delta F}{\delta \varpi_{3n}} = \eta'_{3n}, \dots \quad (14.)$$

and

$$\frac{\delta F}{\delta \eta_1} = -\frac{\delta T}{\delta \eta_1}, \dots, \frac{\delta F}{\delta \eta_{3n}} = -\frac{\delta T}{\delta \eta_{3n}}; \dots \quad (15.)$$

and therefore that the general equation (3.) may receive this new transformation,

$$\frac{d\varpi_i}{dt} = \frac{\delta(U - F)}{\delta \eta_i}. \dots \quad (16.)$$

If then we introduce, for abridgement, the following expression H ,

$$H = F - U = F(\varpi_1, \varpi_2, \dots, \varpi_{3n}, \eta_1, \eta_2, \dots, \eta_{3n}) - U(\eta_1, \eta_2, \dots, \eta_{3n}), \dots \quad (17.)$$

we are conducted to this new manner of presenting the differential equations of motion of a system of n points, attracting or repelling one another :

$$\left. \begin{aligned} \frac{d\eta_1}{dt} &= \frac{\delta H}{\delta \varpi_1}; \quad \frac{d\varpi_1}{dt} = -\frac{\delta H}{\delta \eta_1}; \\ \frac{d\eta_2}{dt} &= \frac{\delta H}{\delta \varpi_2}; \quad \frac{d\varpi_2}{dt} = -\frac{\delta H}{\delta \eta_2}; \\ &\dots \\ \frac{d\eta_{3n}}{dt} &= \frac{\delta H}{\delta \varpi_{3n}}; \quad \frac{d\varpi_{3n}}{dt} = -\frac{\delta H}{\delta \eta_{3n}}. \end{aligned} \right\} \quad (A.)$$

In this view, the problem of mathematical dynamics, for a system of n points, is to integrate a system (A.) of $6n$ ordinary differential equations of the first order, between the $6n$ variables $\eta_i \varpi_i$ and the time t ; and the solution of the problem must consist in assigning these $6n$ variables as functions of the time, and of their own initial values, which we may call $e_i p_i$. And all these $6n$ functions, or $6n$ relations to determine them, may be expressed, with perfect generality and rigour, by the method of the former Essay, or by the following simplified process.

Integration of the Equations of Motion, by means of one Principal Function.

4. If we take the variation of the definite integral

$$S = \int_0^t \left(\Sigma \varpi \frac{\delta H}{\delta \varpi} - H \right) dt \dots \quad (18.)$$

without varying t or dt , we find, by the Calculus of Variations,

$$\delta S = \int_0^t \delta S' \cdot dt, \dots \quad (19.)$$

in which

$$S' = \Sigma \varpi \frac{\delta H}{\delta \varpi} - H, \dots \quad (20.)$$

and therefore

$$\delta S' = \Sigma \left(\varpi \delta \frac{\delta H}{\delta \varpi} - \frac{\delta H}{\delta \eta} \delta \eta \right), \quad \dots \dots \dots \dots \dots \dots \quad (21.)$$

that is, by the equations of motion (A.),

$$\delta S' = \Sigma \left(\varpi \delta \frac{d\eta}{dt} + \frac{d\varpi}{dt} \delta \eta \right) = \frac{d}{dt} \Sigma \varpi \delta \eta; \quad \dots \dots \dots \dots \dots \quad (22.)$$

the variation of the integral S is therefore

$$\delta S = \Sigma (\varpi \delta \eta - p \delta e), \quad \dots \dots \dots \dots \dots \dots \quad (23.)$$

(p and e being still initial values,) and it decomposes itself into the following $6n$ expressions, when S is considered as a function of the $6n$ quantities $\eta_i e_i$, (involving also the time,)

$$\left. \begin{array}{l} \varpi_1 = \frac{\delta S}{\delta \eta_1}; \quad p_1 = - \frac{\delta S}{\delta e_1}; \\ \varpi_2 = \frac{\delta S}{\delta \eta_2}; \quad p_2 = - \frac{\delta S}{\delta e_2}; \\ \dots \dots \dots \\ \varpi_{3n} = \frac{\delta S}{\delta \eta_{3n}}; \quad p_{3n} = - \frac{\delta S}{\delta e_{3n}}; \end{array} \right\} \quad \dots \dots \dots \dots \dots \quad (B.)$$

which are evidently forms for the sought integrals of the $6n$ differential equations of motion (A.), containing only one unknown function S. The difficulty of mathematical dynamics is therefore reduced to the search and study of this one function S, which may for that reason be called the **PRINCIPAL FUNCTION** of motion of a system.

This function S was introduced in the first Essay under the form

$$S = \int_0^t (T + U) dt,$$

the symbols T and U having in this form their recent meanings; and it is worth observing, that when S is expressed by this definite integral, the conditions for its variation vanishing (if the final and initial coordinates and the time be given) are precisely the differential equations of motion (3.), under the forms assigned by **LAGRANGE**. The variation of this definite integral S has therefore the double property, of giving the differential equations of motion for any transformed coordinates when the extreme positions are regarded as fixed, and of giving the integrals of those differential equations when the extreme positions are treated as varying.

5. Although the function S seems to deserve the name here given it of *Principal Function*, as serving to express, in what appears the simplest way, the integrals of the equations of motion, and the differential equations themselves; yet the same analysis conducts to other functions, which also may be used to express the integrals of the same equations. Thus, if we put

$$Q = \int_0^t \left(- \Sigma \eta \frac{\delta H}{\delta \eta} + H \right) dt, \quad \dots \dots \dots \dots \dots \quad (24.)$$

and take the variation of this integral Q without varying t or dt , we find, by a similar process,

$$\delta Q = \Sigma (\eta \delta \varpi - e \delta p); \quad \dots \dots \dots \dots \dots \quad (25.)$$

so that if we consider Q as a function of the $6n$ quantities ϖ_i p_i and of the time, we shall have $6n$ expressions

$$\eta_i = + \frac{\delta Q}{\delta \varpi_i}, \quad e_i = - \frac{\delta Q}{\delta p_i}, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (26.)$$

which are other forms for the integrals of the equations of motion (A.), involving the function Q instead of S . We might also employ the integral

$$V = \int_0^t \Sigma \cdot \varpi \frac{\delta H}{\delta \varpi} dt = \Sigma \int_e^* \varpi d\eta, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (27.)$$

which was called the *Characteristic Function* in the former Essay, and of which, when considered as a function of the $6n+1$ quantities η_i e_i H , the variation is

$$\delta V = \Sigma (\varpi \delta \eta - p \delta e) + t \delta H. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (28.)$$

And all these functions S , Q , V , are connected in such a way, that the forms and properties of any one may be deduced from those of any other.

Investigation of a Pair of Partial Differential Equations of the first Order, which the Principal Function must satisfy.

6. In forming the variation (23.), or the partial differential coefficients (B.), of the Principal Function S , the variation of the time was omitted; but it is easy to calculate the coefficient $\frac{\delta S}{\delta t}$ corresponding to this variation, since the evident equation

$$\frac{\delta S}{\delta t} = \frac{\delta S}{\delta t} + \Sigma \frac{\delta S}{\delta \eta} \frac{d\eta}{dt} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (29.)$$

gives, by (20.), and by (A.), (B.),

$$\frac{\delta S}{\delta t} = S' - \Sigma \cdot \varpi \frac{\delta H}{\delta \varpi} = - H. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (30.)$$

It is evident also that this coefficient, or the quantity $-H$, is constant, so as not to alter during the motion of the system; because the differential equations of motion (A.) give

$$\frac{dH}{dt} = \Sigma \left(\frac{\delta H}{\delta \eta} \frac{d\eta}{dt} + \frac{\delta H}{\delta \varpi} \frac{d\varpi}{dt} \right) = 0. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (31.)$$

If, therefore, we attend to the equation (17.), and observe that the function F is necessarily rational and integer and homogeneous of the second dimension with respect to the quantities ϖ_i , we shall perceive that the principal function S must satisfy the two following equations between its partial differential coefficients of the first order, which offer the chief means of discovering its form:

$$\left. \begin{aligned} \frac{\delta S}{\delta t} + F \left(\frac{\delta S}{\delta \eta_1}, \frac{\delta S}{\delta \eta_2}, \dots, \frac{\delta S}{\delta \eta_{3n}}, \eta_1, \eta_2, \dots, \eta_{3n} \right) &= U(\eta_1, \eta_2, \dots, \eta_{3n}), \\ \frac{\delta S}{\delta t} + F \left(\frac{\delta S}{\delta e_1}, \frac{\delta S}{\delta e_2}, \dots, \frac{\delta S}{\delta e_{3n}}, e_1, e_2, \dots, e_{3n} \right) &= U(e_1, e_2, \dots, e_{3n}). \end{aligned} \right\} (C.)$$

Reciprocally, if the form of S be known, the forms of these equations (C.) can be deduced from it, by elimination of the quantities e or η between the expressions of its partial differential coefficients; and thus we can return from the principal function S to the functions F and U , and consequently to the expression H , and the equations of motion (A.).

Analogous remarks apply to the functions Q and V , which must satisfy the partial differential equations,

$$\left. \begin{aligned} -\frac{\partial Q}{\partial t} + F\left(\varpi_1, \varpi_2, \dots, \varpi_{3n}, \frac{\partial Q}{\partial \varpi_1}, \frac{\partial Q}{\partial \varpi_2}, \dots, \frac{\partial Q}{\partial \varpi_{3n}}\right) &= U\left(\frac{\partial Q}{\partial \varpi_1}, \frac{\partial Q}{\partial \varpi_2}, \dots, \frac{\partial Q}{\partial \varpi_{3n}}\right), \\ -\frac{\partial Q}{\partial t} + F\left(p_1, p_2, \dots, p_{3n}, -\frac{\partial Q}{\partial p_1}, -\frac{\partial Q}{\partial p_2}, \dots, -\frac{\partial Q}{\partial p_{3n}}\right) &= U\left(-\frac{\partial Q}{\partial p_1}, -\frac{\partial Q}{\partial p_2}, \dots, -\frac{\partial Q}{\partial p_{3n}}\right), \end{aligned} \right\} \quad (32.)$$

and

$$\left. \begin{aligned} F\left(\frac{\partial V}{\partial \eta_1}, \frac{\partial V}{\partial \eta_2}, \dots, \frac{\partial V}{\partial \eta_{3n}}, \eta_1, \eta_2, \dots, \eta_{3n}\right) &= H + U(\eta_1, \eta_2, \dots, \eta_{3n}), \\ F\left(\frac{\partial V}{\partial e_1}, \frac{\partial V}{\partial e_2}, \dots, \frac{\partial V}{\partial e_{3n}}, e_1, e_2, \dots, e_{3n}\right) &= H + U(e_1, e_2, \dots, e_{3n}). \end{aligned} \right\} \quad \dots \quad (33.)$$

General Method of improving an approximate Expression for the Principal Function in any Problem of Dynamics.

7. If we separate the principal function S into any two parts,

$$S_1 + S_2 = S, \quad \dots \quad (34.)$$

and substitute their sum for S in the first equation (C.), the function F , from its rational and integer and homogeneous form and dimension, may be expressed in this new way,

$$\left. \begin{aligned} F\left(\frac{\partial S}{\partial \eta_1}, \dots, \frac{\partial S}{\partial \eta_{3n}}, \eta_1, \dots, \eta_{3n}\right) &= F\left(\frac{\partial S_1}{\partial \eta_1}, \dots, \frac{\partial S_1}{\partial \eta_{3n}}, \eta_1, \dots, \eta_{3n}\right) \\ &+ F'\left(\frac{\partial S_1}{\partial \eta_1}\right) \frac{\partial S_2}{\partial \eta_1} + \dots + F'\left(\frac{\partial S_1}{\partial \eta_{3n}}\right) \frac{\partial S_2}{\partial \eta_{3n}} + F\left(\frac{\partial S_2}{\partial \eta_1}, \dots, \frac{\partial S_2}{\partial \eta_{3n}}, \eta_1, \dots, \eta_{3n}\right) \\ &= F\left(\frac{\partial S_1}{\partial \eta_1}, \dots, \frac{\partial S_1}{\partial \eta_{3n}}, \eta_1, \dots, \eta_{3n}\right) - F\left(\frac{\partial S_2}{\partial \eta_1}, \dots, \frac{\partial S_2}{\partial \eta_{3n}}, \eta_1, \dots, \eta_{3n}\right) \\ &+ F'\left(\frac{\partial S_1}{\partial \eta_1}\right) \frac{\partial S_2}{\partial \eta_1} + \dots + F'\left(\frac{\partial S_1}{\partial \eta_{3n}}\right) \frac{\partial S_2}{\partial \eta_{3n}}, \end{aligned} \right\}. \quad (35.)$$

because

$$F\left(\frac{\partial S_1}{\partial \eta_i}\right) = F'\left(\frac{\partial S_1}{\partial \eta_i}\right) - F'\left(\frac{\partial S_2}{\partial \eta_i}\right), \quad \dots \quad (36.)$$

and

$$\Sigma \cdot F'\left(\frac{\partial S_2}{\partial \eta_i}\right) \frac{\partial S_2}{\partial \eta_i} = 2 F\left(\frac{\partial S_2}{\partial \eta_1}, \dots, \frac{\partial S_2}{\partial \eta_{3n}}, \eta_1, \dots, \eta_{3n}\right); \quad \dots \quad (37.)$$

and since, by (A.) and (B.),

$$\mathbf{F}'\left(\frac{\delta S}{\delta \eta_i}\right) = \mathbf{F}'(\varpi_i) = \frac{\delta H}{\delta \varpi_i} = \frac{d \eta_i}{dt}, \quad \dots \quad (38.)$$

we easily transform the first equation (C.) to the following,

$$\frac{dS_2}{dt} = -\frac{\delta S_1}{\delta t} + U(\eta_1, \dots, \eta_{3n}) - F\left(\frac{\delta S_1}{\delta \eta_1}, \dots, \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots, \eta_{3n}\right) + F\left(\frac{\delta S_2}{\delta \eta_1}, \dots, \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \dots, \eta_{3n}\right), \quad (D.)$$

which gives rigorously

$$S_2 = \left. \begin{aligned} & \int_0^t \left\{ -\frac{\delta S_1}{\delta t} + U(\eta_1, \dots, \eta_{3n}) - F\left(\frac{\delta S_1}{\delta \eta_1}, \dots, \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots, \eta_{3n}\right) \right\} dt \\ & + \int_0^t F\left(\frac{\delta S_2}{\delta \eta_1}, \dots, \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \dots, \eta_{3n}\right) dt, \end{aligned} \right\} . \quad (E.)$$

supposing only that the two parts S_1, S_2 , like the whole principal function S , are chosen so as to vanish with the time.

This general and rigorous transformation offers a general method of improving an approximate expression for the principal function S , in any problem of dynamics. For if the part S_1 be such an approximate expression, then the remaining part S_2 will be small; and the homogeneous function F involving the squares and products of the coefficients of this small part, in the second definite integral (E.), will be in general also small, and of a higher order of smallness; we may therefore in general neglect this second definite integral, in passing to a second approximation, and may in general improve a first approximate expression S_1 by adding to it the following correction,

$$\Delta S_1 = \int_0^t \left\{ -\frac{\delta S_1}{\delta t} + U(\eta_1, \dots, \eta_{3n}) - F\left(\frac{\delta S_1}{\delta \eta_1}, \dots, \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots, \eta_{3n}\right) \right\} dt; \quad (F.)$$

in calculating which definite integral we may employ the following approximate forms for the integrals of the equations of motion,

$$p_1 = -\frac{\delta S_1}{\delta e_1}, p_2 = -\frac{\delta S_1}{\delta e_2}, \dots p_{3n} = -\frac{\delta S_1}{\delta e_{3n}}, \dots \dots \dots \dots \dots \quad (39.)$$

expressing first, by these, the variables η_i as functions of the time and of the $6n$ constants e_i, p_i , and then eliminating, after the integration, the $3n$ quantities p_i , by the same approximate forms. And when an improved expression, or second approximate value $S_1 + \Delta S_1$, for the principal function S , has been thus obtained, it may be substituted in like manner for the first approximate value S_1 , so as to obtain a still closer approximation, and the process may be repeated indefinitely.

An analogous process applies to the indefinite improvement of a first approximate expression for the function Q or V .

*Rigorous Theory of Perturbations, founded on the Properties of the Disturbing Part
of the whole Principal Function.*

8. If we separate the expression H (17.) into any two parts of the same kind,

$$H_1 + H_2 = H, \quad \dots \quad (40.)$$

in which

$$H_1 = F_1(\varpi_1, \varpi_2, \dots, \varpi_{3n}, \eta_1, \eta_2, \dots, \eta_{3n}) - U_1(\eta_1, \eta_2, \dots, \eta_{3n}), \dots \quad (41)$$

and

$$H_2 = F_2(\varpi_1, \varpi_2, \dots, \varpi_{3n}, \eta_1, \eta_2, \dots, \eta_{3n}) - U_2(\eta_1, \eta_2, \dots, \eta_{3n}), \dots \quad (42)$$

the functions F_1 F_2 U_1 U_2 being such that

$$F_1 + F_2 = F, \quad U_1 + U_2 = U; \dots \quad (43)$$

the differential equations of motion (A.) will take this form,

$$\frac{d\eta_i}{dt} = \frac{\delta H_1}{\delta \varpi_i} + \frac{\delta H_2}{\delta \varpi_i}, \quad \frac{d\varpi_i}{dt} = -\frac{\delta H_1}{\delta \eta_i} - \frac{\delta H_2}{\delta \eta_i}, \dots \quad (G.)$$

and if the part H_2 and its coefficients be small, they will not differ much from these other differential equations,

$$\frac{d\eta_i}{dt} = \frac{\delta H_1}{\delta \varpi_i}, \quad \frac{d\varpi_i}{dt} = -\frac{\delta H_1}{\delta \eta_i}; \dots \quad (H.)$$

so that the rigorous integrals of the latter system will be approximate integrals of the former. Whenever then, by a proper choice of the predominant term H_1 , a system of $6n$ equations such as (H.) has been formed and rigorously integrated, giving expressions for the $6n$ variables η_i ϖ_i as functions of the time t , and of their own initial values $e_i p_i$, which may be thus denoted :

$$\eta_i = \phi_i(t, e_1, e_2, \dots, e_{3n}, p_1, p_2, \dots, p_{3n}), \dots \quad (44)$$

and

$$\varpi_i = \psi_i(t, e_1, e_2, \dots, e_{3n}, p_1, p_2, \dots, p_{3n}); \dots \quad (45)$$

the simpler motion thus defined by the rigorous integrals of (H.) may be called the *undisturbed motion* of the proposed system of n points, and the more complex motion expressed by the rigorous integrals of (G.) may be called by contrast the *disturbed motion* of that system; and to pass from the one to the other, may be called a *Problem of Perturbation*.

9. To accomplish this passage, let us observe that the differential equations of undisturbed motion (H.), being of the same form as the original equations (A.), may have their integrals similarly expressed, that is, as follows :

$$\varpi_i = \frac{\delta S_1}{\delta \eta_i}, \quad p_i = -\frac{\delta S_1}{\delta e_i}, \dots \quad (I.)$$

S_1 being here the *principal function of undisturbed motion*, or the definite integral

$$S_1 = \int_0^t \left(\sum \varpi_i \frac{\delta H_1}{\delta \varpi_i} - H_1 \right) dt, \dots \quad (46.)$$

considered as a function of the time and of the quantities η_i e_i . In like manner if we represent by $S_1 + S_2$ the whole principal function of disturbed motion, the rigorous integrals of (G.) may be expressed by (B.), as follows :

$$\varpi_i = \frac{\delta S_1}{\delta \eta_i} + \frac{\delta S_2}{\delta \eta_i}, \quad p_i = -\frac{\delta S_1}{\delta e_i} - \frac{\delta S_2}{\delta e_i}, \dots \quad (K.)$$

Comparing the forms (44.) with the second set of equations (I.) for the integrals of undisturbed motion, we find that the following relations between the functions $\varphi_i S_1$ must be rigorously and *identically* true :

$$\eta_i = \varphi_i \left(t, e_1, e_2, \dots e_{3n}, -\frac{\delta S_1}{\delta e_1}, -\frac{\delta S_1}{\delta e_2}, \dots -\frac{\delta S_1}{\delta e_{3n}} \right); \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (47.)$$

and therefore, by (K.), that the integrals of disturbed motion may be put under the following forms,

$$\eta_i = \varphi_i \left(t, e_1, e_2, \dots e_{3n}, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, \dots p_{3n} + \frac{\delta S_2}{\delta e_{3n}} \right). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (L.)$$

We may therefore calculate rigorously the disturbed variables η_i by the rules of undisturbed motion (44.), if without altering the time t , or the initial values e_i of those variables, which determine the initial configuration, we alter (in general) the initial velocities and directions, by adding to the elements p_i the following perturbational terms,

$$\Delta p_1 = \frac{\delta S_2}{\delta e_1}, \quad \Delta p_2 = \frac{\delta S_2}{\delta e_2}, \dots \quad \Delta p_{3n} = \frac{\delta S_2}{\delta e_{3n}}; \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (M.)$$

a remarkable result, which includes the whole theory of perturbation. We might deduce from it the differential coefficients η'_p or the connected quantities ϖ_p , which determine the disturbed directions and velocities of motion at any time t ; but a similar reasoning gives at once the general expression,

$$\varpi_i = \frac{\delta S_2}{\delta \eta_i} + \psi_i \left(t, e_1, e_2, \dots e_{3n}, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, \dots p_{3n} + \frac{\delta S_2}{\delta e_{3n}} \right), \quad \dots \quad (N.)$$

implying, that after altering the initial velocities and directions or the elements p_i as before, by the perturbational terms (M.), we may then employ the rules of undisturbed motion (45.) to calculate the velocities and directions at the time t , or the varying quantities ϖ_p , if we finally apply to these quantities thus calculated the following new corrections for perturbation :

$$\Delta \varpi_1 = \frac{\delta S_2}{\delta \eta_1}, \quad \Delta \varpi_2 = \frac{\delta S_2}{\delta \eta_2}, \dots \quad \Delta \varpi_{3n} = \frac{\delta S_2}{\delta \eta_{3n}}. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (O.)$$

Approximate expressions deduced from the foregoing rigorous Theory.

10. The foregoing theory gives indeed rigorous expressions for the perturbations, in passing from the simpler motion (H.) or (I.) to the more complex motion (G.) or (K.): but it may seem that these expressions are of little use, because they involve an unknown *disturbing function* S_2 , (namely, the perturbational part of the whole principal function S), and also unknown or disturbed coordinates or marks of position η_i . However, it was lately shown that whenever a first approximate form for the principal function S , such as here the principal function S_1 of undisturbed motion, has been found, the correction S_2 can in general be assigned, with an indefinitely increasing

accuracy; and since the perturbations (M.) and (O.) involve the disturbed coordinates η_i only as they enter into the coefficients of this small disturbing function S_2 , it is evidently permitted to substitute for these coordinates, at first, their undisturbed values, and then to correct the results by substituting more accurate expressions.

11. The function S_1 of undisturbed motion must satisfy rigorously two partial differential equations of the form (C.), namely,

$$\left. \begin{aligned} \frac{\delta S_1}{\delta t} + F_1 \left(\frac{\delta S_1}{\delta \eta_1}, \dots, \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots, \eta_{3n} \right) &= U_1(\eta_1, \dots, \eta_{3n}), \\ \frac{\delta S_1}{\delta t} + F_1 \left(\frac{\delta S_1}{\delta e_1}, \dots, \frac{\delta S_1}{\delta e_{3n}}, e_1, \dots, e_{3n} \right) &= U_1(e_1, \dots, e_{3n}); \end{aligned} \right\} \quad \dots \quad (P.)$$

and therefore, by (D.), the disturbing function S_2 must satisfy rigorously the following other condition :

$$\frac{d S_2}{d t} = U_2(\eta_1, \dots, \eta_{3n}) - F_2 \left(\frac{\delta S_1}{\delta \eta_1}, \dots, \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots, \eta_{3n} \right) + F \left(\frac{\delta S_2}{\delta \eta_1}, \dots, \frac{\delta S_2}{\delta \eta_{3n}}, \eta_1, \dots, \eta_{3n} \right), \quad (Q.)$$

and may, on account of the homogeneity and dimension of F , be approximately expressed as follows :

$$S_2 = \int_0^t \left\{ U_2(\eta_1, \dots, \eta_{3n}) - F_2 \left(\frac{\delta S_1}{\delta \eta_1}, \dots, \frac{\delta S_1}{\delta \eta_{3n}}, \eta_1, \dots, \eta_{3n} \right) \right\} dt, \quad \dots \quad (R.)$$

or thus, by (I.),

$$S_2 = \int_0^t \left\{ U_2(\eta_1, \dots, \eta_{3n}) - F_2(\varpi_1, \dots, \varpi_{3n}, \eta_1, \dots, \eta_{3n}) \right\} dt, \quad \dots \quad (S.)$$

that is, by (42.),

$$S_2 = - \int_0^t H_2 dt. \quad \dots \quad (T.)$$

In this expression, H_2 is given immediately as a function of the varying quantities $\eta_i \varpi_i$, but it may be considered in the same order of approximation as a known function of their initial values $e_i p_i$ and of the time t , obtained by substituting for $\eta_i \varpi_i$ their undisturbed values (44.) (45.) as functions of those quantities; its variation may therefore be expressed in either of the two following ways :

$$\delta H_2 = \Sigma \left(\frac{\delta H_2}{\delta \eta} \delta \eta + \frac{\delta H_2}{\delta \varpi} \delta \varpi \right), \quad \dots \quad (48.)$$

or

$$\delta H_2 = \Sigma \left(\frac{\delta H_2}{\delta e} \delta e + \frac{\delta H_2}{\delta p} \delta p \right) + \frac{\delta H_2}{\delta t} \delta t. \quad \dots \quad (49.)$$

Adopting the latter view, and effecting the integration (T.) with respect to the time, by treating the elements $e_i p_i$ as constant, we are afterwards to substitute for the quantities p_i their undisturbed expressions (39.) or (I.), and then we find for the variation of the disturbing function S_2 the expression

$$\delta S_2 = - H_2 \delta t + \Sigma \left(- \delta e \int_0^t \frac{\delta H_2}{\delta e} dt + \delta \frac{\delta S_1}{\delta e} \int_0^t \frac{\delta H_2}{\delta p} dt \right), \quad \dots \quad (50.)$$

which enables us to transform the perturbational terms (M.) (O.) into the following approximate forms :

$$\Delta p_i = - \int_0^t \frac{\delta H_2}{\delta e_i} dt + \Sigma \cdot \frac{\delta^2 S_1}{\delta e \delta e_i} \int_0^t \frac{\delta H_2}{\delta p} dt, \quad \dots \dots \dots \quad (U.)$$

and

$$\Delta \varpi_i = \Sigma \cdot \frac{\delta^2 S_1}{\delta e \delta \eta_i} \int_0^t \frac{\delta H_2}{\delta p} dt, \quad \dots \dots \dots \quad (V.)$$

containing only functions and quantities which may be regarded as given, by the theory of undisturbed motion.

12. In the same order of approximation, if the variation of the expression (44.) for an undisturbed coordinate η_i be thus denoted,

$$\delta \eta_i = \frac{\delta \eta_i}{\delta t} \delta t + \Sigma \left(\frac{\delta \eta_i}{\delta e} \delta e + \frac{\delta \eta_i}{\delta p} \delta p \right), \quad \dots \dots \dots \quad (51.)$$

the perturbation of that coordinate may be expressed as follows :

$$\Delta \eta_i = \Sigma \cdot \frac{\delta \eta_i}{\delta p} \Delta p; \quad \dots \dots \dots \quad (W.)$$

that is, by (U.),

$$\left. \begin{aligned} \Delta \eta_i = & - \frac{\delta \eta_i}{\delta p_1} \int_0^t \frac{\delta H_2}{\delta e_1} dt - \frac{\delta \eta_i}{\delta p_2} \int_0^t \frac{\delta H_2}{\delta e_2} dt - \dots - \frac{\delta \eta_i}{\delta p_{3n}} \int_0^t \frac{\delta H_2}{\delta e_{3n}} dt \\ & + \left(\frac{\delta \eta_i}{\delta p_1} \frac{\delta^2 S_1}{\delta e_1^2} + \frac{\delta \eta_i}{\delta p_2} \frac{\delta^2 S_1}{\delta e_1 \delta e_2} + \dots + \frac{\delta \eta_i}{\delta p_{3n}} \frac{\delta^2 S_1}{\delta e_1 \delta e_{3n}} \right) \int_0^t \frac{\delta H_2}{\delta p_1} dt \\ & + \dots \\ & + \left(\frac{\delta \eta_i}{\delta p_1} \frac{\delta^2 S_1}{\delta e_{3n} \delta e_1} + \frac{\delta \eta_i}{\delta p_2} \frac{\delta^2 S_1}{\delta e_{3n} \delta e_2} + \dots + \frac{\delta \eta_i}{\delta p_{3n}} \frac{\delta^2 S_1}{\delta e_{3n}^2} \right) \int_0^t \frac{\delta H_2}{\delta p_{3n}} dt. \end{aligned} \right\} \quad (52.)$$

Besides, the identical equation (47.) gives

$$\frac{\delta \eta_i}{\delta e_k} = \frac{\delta \eta_i}{\delta p_1} \frac{\delta^2 S_1}{\delta e_k \delta e_1} + \frac{\delta \eta_i}{\delta p_2} \frac{\delta^2 S_1}{\delta e_k \delta e_2} + \dots + \frac{\delta \eta_i}{\delta p_{3n}} \frac{\delta^2 S_1}{\delta e_k \delta e_{3n}}; \quad \dots \dots \quad (53.)$$

the expression (52.) may therefore be thus abridged,

$$\left. \begin{aligned} \Delta \eta_i = & - \frac{\delta \eta_i}{\delta p_1} \int_0^t \frac{\delta H_2}{\delta e_1} dt - \dots - \frac{\delta \eta_i}{\delta p_{3n}} \int_0^t \frac{\delta H_2}{\delta e_{3n}} dt \\ & + \frac{\delta \eta_i}{\delta e_1} \int_0^t \frac{\delta H_2}{\delta p_1} dt + \dots + \frac{\delta \eta_i}{\delta e_{3n}} \int_0^t \frac{\delta H_2}{\delta p_{3n}} dt, \end{aligned} \right\} \quad \dots \dots \quad (X.)$$

and shows that instead of the rigorous perturbational terms (M.) we may approximately employ the following,

$$\Delta p_i = - \int_0^t \frac{\delta H_2}{\delta e_i} dt, \quad \dots \dots \dots \quad (Y.)$$

in order to calculate the disturbed configuration at any time t by the rules of undis-

turbed motion, provided that besides thus altering the initial velocities and directions we alter also the initial configuration, by the formula

$$\Delta e_i = \int_0^t \frac{\delta H_2}{\delta p_i} dt. \quad \dots \dots \dots \dots \dots \quad (Z.)$$

It would not be difficult to calculate, in like manner, approximate expressions for the disturbed directions and velocities at any time t ; but it is better to resume, in another way, the rigorous problem of perturbation.

Other Rigorous Theory of Perturbation, founded on the properties of the disturbing part of the constant of living force, and giving formulae for the Variation of Elements more analogous to those already known.

13. Suppose that the theory of undisturbed motion has given the $6n$ constants $e_i p_i$ or any combinations of these, z_1, z_2, \dots, z_{6n} , as functions of the $6n$ variables $\eta_i \omega_i$ and of the time t , which may be thus denoted:

$$z_i = \chi_i(t, \eta_1, \eta_2, \dots, \eta_{3n}, \omega_1, \omega_2, \dots, \omega_{3n}), \quad \dots \dots \dots \quad (54.)$$

and which give reciprocally expressions for the variables $\eta_i \omega_i$ in terms of these elements and of the time, analogous to (44.) and (45.), and capable of being denoted similarly,

$$\eta_i = \varphi_i(t, z_1, z_2, \dots, z_{6n}), \quad \omega_i = \psi_i(t, z_1, z_2, \dots, z_{6n}); \quad \dots \dots \quad (55.)$$

then, the total differential coefficient of every such element or function z_i , taken with respect to the time, (both as it enters explicitly and implicitly into the expressions (54.),) must vanish in the undisturbed motion; so that, by the differential equations of such motion (H_1), the following general relation must be rigorously and *identically* true :

$$0 = \frac{\delta z_i}{\delta t} + \Sigma \left(\frac{\delta z_i}{\delta \eta} \frac{\delta H_1}{\delta \omega} - \frac{\delta z_i}{\delta \omega} \frac{\delta H_1}{\delta \eta} \right). \quad \dots \dots \dots \quad (56.)$$

In passing to disturbed motion, if we retain the equation (54.) as a *definition* of the quantity z_i , that quantity will no longer be constant, but it will continue to satisfy the inverse relations (55.), and may be called, by analogy, a *varying element* of the motion; and its total differential coefficient, taken with respect to the time, may, by the identical equation (56.), and by the differential equations of disturbed motion (G), be rigorously expressed as follows :

$$\frac{d z_i}{d t} = \Sigma \left(\frac{\delta z_i}{\delta \eta} \frac{\delta H_2}{\delta \omega} - \frac{\delta z_i}{\delta \omega} \frac{\delta H_2}{\delta \eta} \right). \quad \dots \dots \dots \quad (A^1.)$$

14. This result ($A^1.$) contains the whole theory of the gradual variation of the elements of disturbed motion of a system; but it may receive an advantageous transformation, by the substitution of the expressions (55.) for the variables $\eta_i \omega_i$ as functions of the time and of the elements; since it will thus conduct to a system of $6n$

rigorous and ordinary differential equations of the first order between those varying elements and the time. Expressing, therefore, the quantity H_2 as a function of these latter variables, its variation δH_2 takes this new form,

$$\delta H_2 = \Sigma \cdot \frac{\delta H_2}{\delta x} \delta x + \frac{\delta H_2}{\delta t} \delta t, \quad \dots \dots \dots \quad (57.)$$

and gives, by comparison with the form (48.), and by (54.),

$$\frac{\delta H_2}{\delta \eta_r} = \Sigma \cdot \frac{\delta H_2}{\delta x} \frac{\delta x}{\delta \eta_r}; \quad \frac{\delta H_2}{\delta \varpi_r} = \Sigma \cdot \frac{\delta H_2}{\delta x} \frac{\delta x}{\delta \varpi_r}; \quad \dots \dots \dots \quad (58.)$$

and thus the general equation (A¹) is transformed to the following,

$$\frac{dx_i}{dt} = a_{i,1} \frac{\delta H_2}{\delta x_1} + a_{i,2} \frac{\delta H_2}{\delta x_2} + \dots + a_{i,n} \frac{\delta H_2}{\delta x_n}; \quad \dots \dots \dots \quad (B^1.)$$

in which

$$a_{i,s} = \Sigma \left(\frac{\delta x_i}{\delta \eta} \frac{\delta x_s}{\delta \varpi} - \frac{\delta x_i}{\delta \varpi} \frac{\delta x_s}{\delta \eta} \right); \quad \dots \dots \dots \quad (C^1.)$$

so that it only remains to eliminate the variables $\eta \varpi$ from the expressions of these latter coefficients. Now it is remarkable that this elimination removes the symbol t also, and leaves the coefficients $a_{i,s}$ expressed as functions of the elements x alone, not explicitly involving the time. This general theorem of dynamics, which is, perhaps, a little more extensive than the analogous results discovered by LAGRANGE and by POISSON, since it does not limit the disturbing terms in the differential equations of motion to depend on the configuration only, may be investigated in the following way.

15. The sign of summation Σ in (C¹.), like the same sign in those other analogous equations in which it has already occurred without an index in this Essay, refers not to the expressed indices, such as here i, s , in the quantity to be summed, but to an index which is not expressed, and which may be here called r ; so that if we introduce for greater clearness this variable index and its limits, the expression (C¹.) becomes

$$a_{i,s} = \Sigma_{(r)1}^{s_n} \left(\frac{\delta x_i}{\delta \eta_r} \frac{\delta x_s}{\delta \varpi_r} - \frac{\delta x_i}{\delta \varpi_r} \frac{\delta x_s}{\delta \eta_r} \right); \quad \dots \dots \dots \quad (59.)$$

and its total differential coefficient, taken with respect to the time, may be separated into the two following parts,

$$\left. \begin{aligned} \frac{d}{dt} a_{i,s} &= \Sigma_{(r)1}^{s_n} \left(\frac{\delta x_i}{\delta \eta_r} \frac{d}{dt} \frac{\delta x_s}{\delta \varpi_r} - \frac{\delta x_s}{\delta \eta_r} \frac{d}{dt} \frac{\delta x_i}{\delta \varpi_r} \right) \\ &+ \Sigma_{(r)1}^{s_n} \left(\frac{\delta x_s}{\delta \varpi_r} \frac{d}{dt} \frac{\delta x_i}{\delta \eta_r} - \frac{\delta x_i}{\delta \varpi_r} \frac{d}{dt} \frac{\delta x_s}{\delta \eta_r} \right), \end{aligned} \right\} \quad \dots \dots \dots \quad (60.)$$

which we shall proceed to calculate separately, and then to add them together. By the definition (54.), and the differential equations of disturbed motion (G.),

$$\frac{d}{dt} \frac{\delta x_i}{\delta \varpi_r} = \frac{\delta^2 x_i}{\delta t \delta \varpi_r} + \Sigma_{(u)1}^{s_n} \left\{ \frac{\delta^2 x_i}{\delta \eta_u \delta \varpi_r} \left(\frac{\delta H_1}{\delta \varpi_u} + \frac{\delta H_2}{\delta \varpi_u} \right) - \frac{\delta^2 x_i}{\delta \varpi_u \delta \varpi_r} \left(\frac{\delta H_1}{\delta \eta_u} + \frac{\delta H_2}{\delta \eta_u} \right) \right\}, \quad (61.)$$

in which, by the identical equation (56.),

$$\frac{\delta^2 x_i}{\delta t \delta \varpi_r} = - \frac{\delta}{\delta \varpi_r} \sum_{(u)1}^{3n} \left(\frac{\delta x_i}{\delta \eta_u} \frac{\delta H_1}{\delta \varpi_u} - \frac{\delta x_i}{\delta \varpi_u} \frac{\delta H_1}{\delta \eta_u} \right); \quad \dots \quad (62.)$$

we have therefore

$$\frac{d}{dt} \frac{\delta x_i}{\delta \varpi_r} = \sum_{(u)1}^{3n} \left(\frac{\delta^2 x_i}{\delta \eta_u \delta \varpi_r} \frac{\delta H_2}{\delta \varpi_u} - \frac{\delta^2 x_i}{\delta \varpi_u \delta \varpi_r} \frac{\delta H_2}{\delta \eta_u} + \frac{\delta x_i}{\delta \varpi_u} \frac{\delta^2 H_1}{\delta \eta_u \delta \varpi_r} - \frac{\delta x_i}{\delta \eta_u} \frac{\delta^2 H_1}{\delta \varpi_u \delta \varpi_r} \right), \quad (63.)$$

and $\frac{d}{dt} \frac{\delta x_s}{\delta \varpi_r}$ may be found from this, by merely changing i to s : so that

$$\left. \begin{aligned} \sum_{(r)1}^{3n} \left(\frac{\delta x_i}{\delta \eta_r} \frac{d}{dt} \frac{\delta x_s}{\delta \varpi_r} - \frac{\delta x_s}{\delta \eta_r} \frac{d}{dt} \frac{\delta x_i}{\delta \varpi_r} \right) = \\ \sum_{(r,u)1,1}^{3n,3n} \left\{ \left(\frac{\delta x_s}{\delta \eta_r} \frac{\delta^2 x_i}{\delta \varpi_u \delta \varpi_r} - \frac{\delta x_i}{\delta \eta_r} \frac{\delta^2 x_s}{\delta \varpi_u \delta \varpi_r} \right) \frac{\delta H_2}{\delta \eta_u} + \left(\frac{\delta x_i}{\delta \eta_r} \frac{\delta^2 x_s}{\delta \eta_u \delta \varpi_r} - \frac{\delta x_s}{\delta \eta_r} \frac{\delta^2 x_i}{\delta \eta_u \delta \varpi_r} \right) \frac{\delta H_2}{\delta \varpi_u} \right. \\ \left. + \left(\frac{\delta x_i}{\delta \eta_r} \frac{\delta x_s}{\delta \varpi_u} - \frac{\delta x_s}{\delta \eta_r} \frac{\delta x_i}{\delta \varpi_u} \right) \frac{\delta^2 H_1}{\delta \eta_u \delta \varpi_r} + \left(\frac{\delta x_s}{\delta \eta_r} \frac{\delta x_i}{\delta \eta_u} - \frac{\delta x_i}{\delta \eta_r} \frac{\delta x_s}{\delta \eta_u} \right) \frac{\delta^2 H_1}{\delta \varpi_u \delta \varpi_r} \right\}, \end{aligned} \right\} \quad (64.)$$

and similarly,

$$\left. \begin{aligned} \sum_{(r)1}^{3n} \left(\frac{\delta x_s}{\delta \varpi_r} \frac{d}{dt} \frac{\delta x_i}{\delta \eta_r} - \frac{\delta x_i}{\delta \varpi_r} \frac{d}{dt} \frac{\delta x_s}{\delta \eta_r} \right) = \\ \sum_{(r,u)1,1}^{3n,3n} \left\{ \left(\frac{\delta x_s}{\delta \varpi_r} \frac{\delta^2 x_i}{\delta \eta_u \delta \eta_r} - \frac{\delta x_i}{\delta \varpi_r} \frac{\delta^2 x_s}{\delta \eta_u \delta \eta_r} \right) \frac{\delta H_2}{\delta \varpi_u} + \left(\frac{\delta x_i}{\delta \varpi_r} \frac{\delta^2 x_s}{\delta \varpi_u \delta \eta_r} - \frac{\delta x_s}{\delta \varpi_r} \frac{\delta^2 x_i}{\delta \varpi_u \delta \eta_r} \right) \frac{\delta H_2}{\delta \eta_u} \right. \\ \left. + \left(\frac{\delta x_i}{\delta \varpi_r} \frac{\delta x_s}{\delta \eta_u} - \frac{\delta x_s}{\delta \varpi_r} \frac{\delta x_i}{\delta \eta_u} \right) \frac{\delta^2 H_1}{\delta \varpi_u \delta \eta_r} + \left(\frac{\delta x_s}{\delta \varpi_r} \frac{\delta x_i}{\delta \eta_u} - \frac{\delta x_i}{\delta \varpi_r} \frac{\delta x_s}{\delta \eta_u} \right) \frac{\delta^2 H_1}{\delta \eta_u \delta \eta_r} \right\}. \end{aligned} \right\} \quad (65.)$$

Adding, therefore, the two last expressions, and making the reductions which present themselves, we find, by (60.),

$$\frac{d}{dt} a_{i,s} = \sum_{(u)1}^{3n} \left(A_{i,s}^{(u)} \frac{\delta H_2}{\delta \eta_u} + B_{i,s}^{(u)} \frac{\delta H_2}{\delta \varpi_u} \right), \quad \dots \quad (D^1.)$$

in which

$$\left. \begin{aligned} A_{i,s}^{(u)} = \sum_{(r)1}^{3n} \left(\frac{\delta x_s}{\delta \eta_r} \frac{\delta^2 x_i}{\delta \varpi_u \delta \varpi_r} - \frac{\delta x_i}{\delta \eta_r} \frac{\delta^2 x_s}{\delta \varpi_u \delta \varpi_r} + \frac{\delta x_i}{\delta \varpi_r} \frac{\delta^2 x_s}{\delta \eta_u \delta \eta_r} - \frac{\delta x_s}{\delta \varpi_r} \frac{\delta^2 x_i}{\delta \eta_u \delta \eta_r} \right), \\ B_{i,s}^{(u)} = \sum_{(r)1}^{3n} \left(\frac{\delta x_s}{\delta \varpi_r} \frac{\delta^2 x_i}{\delta \eta_u \delta \eta_r} - \frac{\delta x_i}{\delta \varpi_r} \frac{\delta^2 x_s}{\delta \eta_u \delta \eta_r} + \frac{\delta x_i}{\delta \eta_r} \frac{\delta^2 x_s}{\delta \eta_u \delta \varpi_r} - \frac{\delta x_s}{\delta \eta_r} \frac{\delta^2 x_i}{\delta \eta_u \delta \varpi_r} \right); \end{aligned} \right\} \quad (66.)$$

and since this general form (D¹.) for $\frac{d}{dt} a_{i,s}$ contains no term independent of the disturbing quantities $\frac{\delta H_2}{\delta \eta}, \frac{\delta H_2}{\delta \varpi}$, it is easy to infer from it the important consequence already mentioned, namely, that the coefficients $a_{i,s}$, in the differentials (B¹) of the elements, may be expressed as functions of those elements alone, not explicitly involving the time.

It is evident also, that these coefficients $a_{i,s}$ have the property

$$a_{s,i} = -a_{i,s} \quad \dots \quad (67.)$$

and

the term proportional to $\frac{\delta H_2}{\delta x_i}$ disappears therefore from the expression (B¹) for $\frac{dx_i}{dt}$;

$$\frac{\delta H_2}{\delta x_i} \cdot a_{i,s} \cdot \frac{\delta H_2}{\delta x_s} \text{ in } \frac{\delta H_2}{\delta x_i} \frac{dx_i}{dt},$$

destroys the term

$$\frac{\delta H_2}{\delta x_s} \cdot a_{s,i} \cdot \frac{\delta H_2}{\delta x_i} \text{ in } \frac{\delta H_2}{\delta x_s} \frac{dx_s}{dt},$$

when these terms are added together; we have, therefore,

OR

that is, in taking the first total differential coefficient of the disturbing expression H_2 with respect to the time, the elements may be treated as constant.

Simplification of the differential equations which determine these gradually varying elements, in any problem of Perturbation; and Integration of the simplified equations by means of certain Functions of Elements.

16. The most natural choice of these elements is that which makes them correspond, in undisturbed motion, to the initial quantities $e_i p_i$. These quantities, by the differential equations (H.), may be expressed in undisturbed motion as follows.

$$e_i = \eta_i - \int_0^t \frac{\delta H_1}{\delta \varpi_i} d\tau, \quad p_i = \varpi_i + \int_0^t \frac{\delta H_1}{\delta \eta_i} d\tau; \quad \dots \quad (69.)$$

and if we suppose them found, by elimination, under the forms

$$\left. \begin{aligned} e_i &= \eta_i + \Phi_i(t, \eta_1, \eta_2, \dots, \eta_{3n}, \varpi_1, \varpi_2, \dots, \varpi_{3n}), \\ p_i &= \varpi_i + \Psi_i(t, \eta_1, \eta_2, \dots, \eta_{3n}, \varpi_1, \varpi_2, \dots, \varpi_{3n}), \end{aligned} \right\} \quad \dots \quad (70.)$$

it is easy to see that the following equations must be rigorously and identically true, for all values of η_i , ϖ_i ,

$$\left. \begin{array}{l} 0 = \Phi_i(0, \eta_1, \eta_2, \dots, \eta_{3n}, \varpi_1, \varpi_2, \dots, \varpi_{3n}), \\ 0 = \Psi_i(0, \eta_1, \eta_2, \dots, \eta_{3n}, \varpi_1, \varpi_2, \dots, \varpi_{3n}). \end{array} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (71.)$$

When, therefore, in passing to disturbed motion, we establish the equations of definition,

$$\left. \begin{aligned} x_i &= \eta_i + \Phi_i(t, \eta_1, \eta_2, \dots, \eta_{3n}, \varpi_1, \varpi_2, \dots, \varpi_{3n}), \\ \lambda_i &= \varpi_i + \Psi_i(t, \eta_1, \eta_2, \dots, \eta_{3n}, \varpi_1, \varpi_2, \dots, \varpi_{3n}), \end{aligned} \right\} \quad . \quad (72.)$$

introducing $6n$ varying elements z_i, λ_i , of which the set λ_i would have been represented in our recent notation as follows:

we see that all the partial differential coefficients of the forms $\frac{\partial x_i}{\partial \eta_r}, \frac{\partial x_i}{\partial \omega_r}, \frac{\partial \lambda_i}{\partial \eta_r}, \frac{\partial \lambda_i}{\partial \omega_r}$, vanish when $t = 0$, except the following :

$$\frac{\delta x_i}{\delta n} = 1, \frac{\delta \lambda_i}{\delta \omega_i} = 1; \dots \dots \dots \dots \dots \dots \dots \quad (74.)$$

and, therefore, that when t is made = 0, in the coefficients $a_{i,s}$, (59), all those coefficients vanish, except the following :

$$a_{r,3n+r} = 1, \quad a_{3n+r,r} = -1, \quad \dots, \quad (75.)$$

But it has been proved that these coefficients $a_{i,s}$, when expressed as functions of the elements, do not contain the time explicitly; and the supposition $t = 0$ introduces no relation between those $6n$ elements z_i, λ_i , which still remain independent: the coefficients $a_{i,s}$, therefore, could not acquire the values $1, 0, -1$, by the supposition $t = 0$, unless they had those values constantly, and independently of that supposition. The differential equations of the forms (B¹.), may therefore be expressed, for the present system of varying elements, in the following simpler way:

and an easy verification of these expressions is offered by the formula (E¹), which takes now this form,

$$\Sigma \left(\frac{\delta H_2}{\delta x} \frac{dx}{dt} + \frac{\delta H_2}{\delta \lambda} \frac{d\lambda}{dt} \right) = 0. \quad \dots \quad (H^1)$$

17. The initial values of the varying elements z_i, λ_i are evidently e_i, p_i , by the definitions (72.), and by the identical equations (71.); the problem of integrating rigorously the equations of disturbed motion (G.), between the variables η_i, ϖ_i and the time, or of determining these variables as functions of the time and of their own initial values e_i, p_i , is therefore rigorously transformed into the problem of integrating the equations (G^{1.}), or of determining the $6n$ elements z_i, λ_i as functions of the time and of the same initial values. The chief advantage of this transformation is, that if the perturbations be small, the new variables (namely, the elements,) alter but little: and that, since the new differential equations are of the same form as the old, they may be integrated by a similar method. Considering, therefore, the definite integral

$$\mathbf{E} = \int_s^t \left(\Sigma \cdot \lambda \frac{\partial \mathbf{H}_2}{\partial \lambda} - \mathbf{H}_2 \right) dt, \quad \dots \quad (76.)$$

as a function of the time and of the $6n$ quantities $z_1, z_2, \dots, z_{3n}, e_1, e_2, \dots, e_{3n}$, and observing that its variation, taken with respect to the latter quantities, may be shown by a process similar to that of the fourth number of this Essay to be

$$\delta E = \Sigma (\lambda_i \delta z - p_i \delta e), \dots \dots \dots \dots \dots \dots \quad (I^1)$$

we find that the rigorous integrals of the differential equations (G¹) may be expressed in the following manner:

$$\lambda_i = \frac{\delta E}{\delta z_i}, \quad p_i = -\frac{\delta E}{\delta e_i}, \dots \dots \dots \dots \dots \dots \quad (K^1)$$

in which there enters only one unknown *function of elements* E, to the search and study of which single function the problem of perturbation is reduced by this new method.

We might also have put

$$C = \int_0^t \left(-\Sigma z_i \frac{\delta H_2}{\delta z_i} + H_2 \right) dt, \dots \dots \dots \dots \dots \dots \quad (77)$$

and have considered this definite integral C as a function of the time and of the $6n$ quantities $\lambda_i p_i$; and then we should have found the following other forms for the integrals of the differential equations of varying elements,

$$z_i = +\frac{\delta C}{\delta \lambda_i}, \quad e_i = -\frac{\delta C}{\delta p_i}, \dots \dots \dots \dots \dots \dots \quad (L^1)$$

And each of these *functions of elements*, C and E, must satisfy a certain partial differential equation, analogous to the first equation of each pair mentioned in the sixth number of this Essay, and deduced on similar principles.

18. Thus, it is evident, by the form of the function E, and by the equations (K¹), (G¹), and (76.), that the partial differential coefficient of this function, taken with respect to the time, is

$$\frac{\delta E}{\delta t} = \frac{d E}{d t} - \Sigma z_i \frac{\delta E}{\delta z_i} \frac{d z_i}{d t} = -H_2; \dots \dots \dots \dots \dots \dots \quad (M^1)$$

and therefore that if we separate this function E into any two parts

$$E_1 + E_2 = E, \dots \dots \dots \dots \dots \dots \quad (N^1)$$

and if, for greater clearness, we put the expression H_2 under the form

$$H_2 = H_2(t, z_1, z_2, \dots, z_{3n}, \lambda_1, \lambda_2, \dots, \lambda_{3n}), \dots \dots \dots \dots \dots \dots \quad (O^1)$$

we shall have rigorously the partial differential equation

$$0 = \frac{\delta E_1}{\delta t} + \frac{\delta E_2}{\delta t} + H_2 \left(t, z_1, \dots, z_{3n}, \frac{\delta E_1}{\delta z_1}, \frac{\delta E_2}{\delta z_1}, \dots, \frac{\delta E_1}{\delta z_{3n}}, \frac{\delta E_2}{\delta z_{3n}} \right); \quad (P^1)$$

which gives, approximately, by (G¹) and (K¹), when the part E_2 is small, and when we neglect the squares and products of its partial differential coefficients,

$$0 = \frac{d E_2}{d t} + \frac{\delta E_1}{\delta t} + H_2 \left(t, z_1, \dots, z_{3n}, \frac{\delta E_1}{\delta z_1}, \dots, \frac{\delta E_1}{\delta z_{3n}} \right). \dots \dots \dots \dots \dots \dots \quad (Q^1)$$

Hence, in the same order of approximation, if the part E_1 , like the whole function E , be chosen so as to vanish with the time, we shall have

$$E_2 = - \int_0^t \left\{ \frac{\delta E_1}{\delta t} + H_2 \left(t, x_1, \dots, x_{3n}, \frac{\delta E_1}{\delta x_1}, \dots, \frac{\delta E_1}{\delta x_{3n}} \right) \right\} : \dots \quad (R^1)$$

and thus a first approximate expression E_1 can be successively and indefinitely corrected.

Again, by (L¹.) and (G¹.), and by the definition (77.),

$$\frac{\delta C}{\delta t} = \frac{dC}{dt} - \sum \frac{\delta C}{\delta \lambda} \frac{d\lambda}{dt} = H_2; \quad \dots \quad (S^1)$$

the function C must therefore satisfy rigorously the partial differential equation,

$$\frac{\delta C}{\delta t} = H_2 \left(t, \frac{\delta C}{\delta \lambda_1}, \dots, \frac{\delta C}{\delta \lambda_{3n}}, \lambda_1, \dots, \lambda_{3n} \right); \quad \dots \quad (T^1)$$

and if we put

$$C = C_1 + C_2, \quad \dots \quad (U^1)$$

and suppose that the part C_2 is small, then the rigorous equation

$$\frac{\delta C_1}{\delta t} + \frac{\delta C_2}{\delta t} = H_2 \left(t, \frac{\delta C_1}{\delta \lambda_1} + \frac{\delta C_2}{\delta \lambda_1}, \dots, \frac{\delta C_1}{\delta \lambda_{3n}} + \frac{\delta C_2}{\delta \lambda_{3n}}, \lambda_1, \dots, \lambda_{3n} \right) \quad \dots \quad (V^1)$$

becomes approximately, by (G¹.) and (L¹.),

$$\frac{dC_2}{dt} = - \frac{\delta C_1}{\delta t} + H_2 \left(t, \frac{\delta C_1}{\delta \lambda_1}, \dots, \frac{\delta C_1}{\delta \lambda_{3n}}, \lambda_1, \dots, \lambda_{3n} \right), \quad \dots \quad (W^1)$$

and gives by integration,

$$C_2 = \int_0^t \left\{ - \frac{\delta C_1}{\delta t} + H_2 \left(t, \frac{\delta C_1}{\delta \lambda_1}, \dots, \frac{\delta C_1}{\delta \lambda_{3n}}, \lambda_1, \dots, \lambda_{3n} \right) \right\} dt, \quad \dots \quad (X^1)$$

the parts C_1 and C_2 being supposed to vanish separately when $t = 0$, like the whole function of elements C .

And to obtain such a first approximation, E_1 or C_1 , to either of these two functions of elements E , C , we may change, in the definitions (76.) (77.), the varying elements x , λ , to their initial values e , p , and then eliminate one set of these initial values by the corresponding set of the following approximate equations, deduced from the formulae (G¹.) :

$$x_i = e_i + \int_0^t \frac{\delta H_2}{\delta p_i} dt; \quad \dots \quad (Y^1)$$

and

$$\lambda_i = p_i - \int_0^t \frac{\delta H_2}{\delta e_i} dt. \quad \dots \quad (Z^1)$$

It is easy also to see that these two functions of elements C and E are connected with each other, and with the disturbing function S_2 , so that the form of any one may be deduced from that of any other, when the function S_1 of undisturbed motion is known.

Analogous formulae for the motion of a Single Point.

19. Our general method in dynamics, though intended chiefly for the study of attracting and repelling systems, is not confined to such, but may be used in all questions to which the law of living forces applies. And all the analysis of this Essay, but especially the theory of perturbations, may usefully be illustrated by the following analogous reasonings and results respecting the motion of a single point.

Imagine then such a point, having for its three rectangular coordinates $x y z$, and moving in an orbit determined by three ordinary differential equations of the second order of forms analogous to the equations (2.), namely,

$$x'' = \frac{\delta U}{\delta x}; y'' = \frac{\delta U}{\delta y}; z'' = \frac{\delta U}{\delta z}; \dots \dots \dots \dots \dots \quad (78.)$$

U being any given function of the coordinates not expressly involving the time: and let us establish the following definition, analogous to (4.),

$$T = \frac{1}{2} (x'^2 + y'^2 + z'^2), \dots \dots \dots \dots \dots \quad (79.)$$

$x' y' z'$ being the first, and $x'' y'' z''$ being the second differential coefficients of the coordinates, considered as functions of the time t . If we express, for greater generality or facility, the rectangular coordinates $x y z$ as functions of three other marks of position $\eta_1 \eta_2 \eta_3$, T will become a homogeneous function of the second dimension of their first differential coefficients $\eta'_1 \eta'_2 \eta'_3$ taken with respect to the time; and if we put, for abridgement,

$$\varpi_1 = \frac{\delta T}{\delta \eta'_1}, \varpi_2 = \frac{\delta T}{\delta \eta'_2}, \varpi_3 = \frac{\delta T}{\delta \eta'_3}, \dots \dots \dots \dots \dots \quad (80.)$$

T may be considered also as a function of the form

$$T = F(\varpi_1, \varpi_2, \varpi_3, \eta_1, \eta_2, \eta_3), \dots \dots \dots \dots \dots \quad (81.)$$

which will be homogeneous of the second dimension with respect to $\varpi_1 \varpi_2 \varpi_3$. We may also put, for abridgement,

$$F(\varpi_1, \varpi_2, \varpi_3, \eta_1, \eta_2, \eta_3) - U(\eta_1, \eta_2, \eta_3) = H; \dots \dots \dots \dots \dots \quad (82.)$$

and then, instead of the three differential equations of the second order (78.), we may employ the six following of the first order, analogous to the equations (A.), and obtained by a similar reasoning,

$$\left. \begin{aligned} \frac{d \eta_1}{d t} &= + \frac{\delta H}{\delta \varpi_1}, \quad \frac{d \eta_2}{d t} = + \frac{\delta H}{\delta \varpi_2}, \quad \frac{d \eta_3}{d t} = + \frac{\delta H}{\delta \varpi_3}, \\ \frac{d \varpi_1}{d t} &= - \frac{\delta H}{\delta \eta_1}, \quad \frac{d \varpi_2}{d t} = - \frac{\delta H}{\delta \eta_2}, \quad \frac{d \varpi_3}{d t} = - \frac{\delta H}{\delta \eta_3}. \end{aligned} \right\} \dots \dots \dots \dots \dots \quad (83.)$$

20. The rigorous integrals of these six differential equations may be expressed under the following forms, analogous to (B.),

$$\left. \begin{aligned} \varpi_1 &= \frac{\delta S}{\delta \eta_1}, \quad \varpi_2 = \frac{\delta S}{\delta \eta_2}, \quad \varpi_3 = \frac{\delta S}{\delta \eta_3}, \\ p_1 &= - \frac{\delta S}{\delta e_1}, \quad p_2 = - \frac{\delta S}{\delta e_2}, \quad p_3 = - \frac{\delta S}{\delta e_3}, \end{aligned} \right\} \dots \dots \dots \dots \dots \quad (84.)$$

in which $e_1 e_2 e_3 p_1 p_2 p_3$ are the initial values, or values at the time 0, of $\eta_1 \eta_2 \eta_3$ $\varpi_1 \varpi_2 \varpi_3$; and S is the definite integral

$$S = \int_0^t \left(\varpi_1 \frac{\partial H}{\partial \varpi_1} + \varpi_2 \frac{\partial H}{\partial \varpi_2} + \varpi_3 \frac{\partial H}{\partial \varpi_3} - H \right) dt, \quad \dots \quad (85.)$$

considered as a function of $\eta_1 \eta_2 \eta_3 e_1 e_2 e_3$ and t . The quantity H does not change in the course of the motion, and the function S must satisfy the following pair of partial differential equations of the first order, analogous to the pair (C.),

$$\left. \begin{aligned} \frac{\partial S}{\partial t} + F \left(\frac{\partial S}{\partial \eta_1}, \frac{\partial S}{\partial \eta_2}, \frac{\partial S}{\partial \eta_3}, \eta_1, \eta_2, \eta_3 \right) &= U(\eta_1, \eta_2, \eta_3); \\ \frac{\partial S}{\partial t} + F \left(\frac{\partial S}{\partial e_1}, \frac{\partial S}{\partial e_2}, \frac{\partial S}{\partial e_3}, e_1, e_2, e_3 \right) &= U(e_1, e_2, e_3). \end{aligned} \right\} \quad \dots \quad (86.)$$

This important function S , which may be called the *principal function* of the motion, may hence be rigorously expressed under the following form, obtained by reasonings analogous to those of the seventh number of this Essay :

$$\left. \begin{aligned} S = S_1 + \int_0^t \left\{ - \frac{\partial S_1}{\partial t} + U(\eta_1, \eta_2, \eta_3) - F \left(\frac{\partial S_1}{\partial \eta_1}, \frac{\partial S_1}{\partial \eta_2}, \frac{\partial S_1}{\partial \eta_3}, \eta_1, \eta_2, \eta_3 \right) \right\} dt \\ + \int_0^t F \left(\frac{\partial S_1}{\partial \eta_1} - \frac{\partial S_1}{\partial \eta_1}, \frac{\partial S_1}{\partial \eta_2} - \frac{\partial S_1}{\partial \eta_2}, \frac{\partial S_1}{\partial \eta_3} - \frac{\partial S_1}{\partial \eta_3}, \eta_1, \eta_2, \eta_3 \right) dt; \end{aligned} \right\} \quad (87.)$$

S_1 being any arbitrary function of the same quantities $\eta_1 \eta_2 \eta_3 e_1 e_2 e_3 t$, so chosen as to vanish with the time. And if this arbitrary function S_1 be chosen so as to be a first approximate value of the principal function S , we may neglect, in a second approximation, the second definite integral in (87.).

21. A first approximation of this kind can be obtained, whenever, by separating the expression H , (82.), into a predominant and a smaller part, H_1 and H_2 , and by neglecting the part H_2 , we have changed the differential equations (83.) to others, namely,

$$\left. \begin{aligned} \frac{d \eta_1}{d t} &= \frac{\partial H_1}{\partial \varpi_1}, \quad \frac{d \eta_2}{d t} = \frac{\partial H_1}{\partial \varpi_2}, \quad \frac{d \eta_3}{d t} = \frac{\partial H_1}{\partial \varpi_3}, \\ \frac{d \varpi_1}{d t} &= - \frac{\partial H_1}{\partial \eta_1}, \quad \frac{d \varpi_2}{d t} = - \frac{\partial H_1}{\partial \eta_2}, \quad \frac{d \varpi_3}{d t} = - \frac{\partial H_1}{\partial \eta_3}, \end{aligned} \right\} \quad \dots \quad (88.)$$

and have succeeded in integrating rigorously these simplified equations, belonging to a simpler motion, which may be called the *undisturbed motion* of the point. For the principal function of such undisturbed motion, namely, the definite integral

$$S_1 = \int_0^t \left(\varpi_1 \frac{\partial H_1}{\partial \varpi_1} + \varpi_2 \frac{\partial H_1}{\partial \varpi_2} + \varpi_3 \frac{\partial H_1}{\partial \varpi_3} - H_1 \right) dt, \quad \dots \quad (89.)$$

considered as a function of $\eta_1 \eta_2 \eta_3 e_1 e_2 e_3 t$, will then be an approximate value for the original function of disturbed motion S , which original function corresponds to the more complex differential equations,

$$\left. \begin{aligned} \frac{d\eta_1}{dt} &= \frac{\delta H_1}{\delta \omega_1} + \frac{\delta H_2}{\delta \omega_1}, \frac{d\eta_2}{dt} = \frac{\delta H_1}{\delta \omega_2} + \frac{\delta H_2}{\delta \omega_2}, \frac{d\eta_3}{dt} = \frac{\delta H_1}{\delta \omega_3} + \frac{\delta H_2}{\delta \omega_3}, \\ \frac{d\omega_1}{dt} &= -\frac{\delta H_1}{\delta \eta_1} - \frac{\delta H_2}{\delta \eta_1}, \frac{d\omega_2}{dt} = -\frac{\delta H_1}{\delta \eta_2} - \frac{\delta H_2}{\delta \eta_2}, \frac{d\omega_3}{dt} = -\frac{\delta H_1}{\delta \eta_3} - \frac{\delta H_2}{\delta \eta_3}, \end{aligned} \right\} \quad (90.)$$

The function S_1 of undisturbed motion must satisfy a pair of partial differential equations of the first order, analogous to the pair (86.); and the integrals of undisturbed motion may be represented thus,

$$\left. \begin{aligned} \omega_1 &= \frac{\delta S_1}{\delta \eta_1}, \omega_2 = \frac{\delta S_1}{\delta \eta_2}, \omega_3 = \frac{\delta S_1}{\delta \eta_3}, \\ p_1 &= -\frac{\delta S_1}{\delta e_1}, p_2 = -\frac{\delta S_1}{\delta e_2}, p_3 = -\frac{\delta S_1}{\delta e_3}; \end{aligned} \right\} \quad \dots \dots \dots \dots \quad (91.)$$

while the integrals of disturbed motion may be expressed with equal rigour under the following analogous forms,

$$\left. \begin{aligned} \omega_1 &= \frac{\delta S_1}{\delta \eta_1} + \frac{\delta S_2}{\delta \eta_1}, \omega_2 = \frac{\delta S_1}{\delta \eta_2} + \frac{\delta S_2}{\delta \eta_2}, \omega_3 = \frac{\delta S_1}{\delta \eta_3} + \frac{\delta S_2}{\delta \eta_3}, \\ p_1 &= -\frac{\delta S_1}{\delta e_1} - \frac{\delta S_2}{\delta e_1}, p_2 = -\frac{\delta S_1}{\delta e_2} - \frac{\delta S_2}{\delta e_2}, p_3 = -\frac{\delta S_1}{\delta e_3} - \frac{\delta S_2}{\delta e_3}, \end{aligned} \right\} \quad \dots \quad (92.)$$

if S_2 denote the rigorous correction of S_1 , or the disturbing part of the whole principal function S . And by the foregoing general theory of approximation, this disturbing part or function S_2 may be approximately represented by the definite integral (T.),

$$S_2 = - \int_0^t H_2 dt; \quad \dots \dots \dots \dots \quad (93.)$$

in calculating which definite integral the equations (91.) may be employed.

22. If the integrals of undisturbed motion (91.) have given

$$\left. \begin{aligned} \eta_1 &= \varphi_1(t, e_1, e_2, e_3, p_1, p_2, p_3), \\ \eta_2 &= \varphi_2(t, e_1, e_2, e_3, p_1, p_2, p_3), \\ \eta_3 &= \varphi_3(t, e_1, e_2, e_3, p_1, p_2, p_3), \end{aligned} \right\} \quad \dots \dots \dots \dots \quad (94.)$$

and

$$\left. \begin{aligned} \omega_1 &= \psi_1(t, e_1, e_2, e_3, p_1, p_2, p_3), \\ \omega_2 &= \psi_2(t, e_1, e_2, e_3, p_1, p_2, p_3), \\ \omega_3 &= \psi_3(t, e_1, e_2, e_3, p_1, p_2, p_3), \end{aligned} \right\} \quad \dots \dots \dots \dots \quad (95.)$$

then the integrals of disturbed motion (92.) may be rigorously transformed as follows,

$$\left. \begin{aligned} \eta_1 &= \varphi_1 \left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3} \right), \\ \eta_2 &= \varphi_2 \left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3} \right), \\ \eta_3 &= \varphi_3 \left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3} \right), \end{aligned} \right\} \quad \dots \quad (96.)$$

and

$$\left. \begin{aligned} \varpi_1 &= \frac{\delta S_2}{\delta \eta_1} + \psi_1 \left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3} \right), \\ \varpi_2 &= \frac{\delta S_2}{\delta \eta_2} + \psi_2 \left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3} \right), \\ \varpi_3 &= \frac{\delta S_2}{\delta \eta_3} + \psi_3 \left(t, e_1, e_2, e_3, p_1 + \frac{\delta S_2}{\delta e_1}, p_2 + \frac{\delta S_2}{\delta e_2}, p_3 + \frac{\delta S_2}{\delta e_3} \right), \end{aligned} \right\} . \quad (97.)$$

S_2 being here the rigorous disturbing function. And the perturbations of position, at any time t , may be approximately expressed by the following formula,

$$\left. \begin{aligned} \Delta \eta_1 &= \frac{\delta \eta_1}{\delta e_1} \int_0^t \frac{\delta H_2}{\delta p_1} dt + \frac{\delta \eta_1}{\delta e_2} \int_0^t \frac{\delta H_2}{\delta p_2} dt + \frac{\delta \eta_1}{\delta e_3} \int_0^t \frac{\delta H_2}{\delta p_3} dt \\ &\quad - \frac{\delta \eta_1}{\delta p_1} \int_0^t \frac{\delta H_2}{\delta e_1} dt - \frac{\delta \eta_1}{\delta p_2} \int_0^t \frac{\delta H_2}{\delta e_2} dt - \frac{\delta \eta_1}{\delta p_3} \int_0^t \frac{\delta H_2}{\delta e_3} dt, \end{aligned} \right\} . \quad (98.)$$

together with two similar formulae for the perturbations of the two other coordinates, or marks of position η_2, η_3 . In these formulæ, the coordinates and H_2 are supposed to be expressed, by the theory of undisturbed motion, as functions of the time t , and of the constants $e_1, e_2, e_3, p_1, p_2, p_3$.

23. Again, if the integrals of undisturbed motion have given, by elimination, expressions for these constants, of the forms

$$\left. \begin{aligned} e_1 &= \eta_1 + \Phi_1(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3), \\ e_2 &= \eta_2 + \Phi_2(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3), \\ e_3 &= \eta_3 + \Phi_3(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3), \end{aligned} \right\} \quad \dots \dots \dots \dots \dots \quad (99).$$

and

$$\left. \begin{aligned} p_1 &= \varpi_1(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3), \\ p_2 &= \varpi_2(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3), \\ p_3 &= \varpi_3(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3); \end{aligned} \right\} \quad \dots \dots \dots \dots \dots \quad (100.)$$

and if, for disturbed motion, we establish the definitions

$$\left. \begin{aligned} z_1 &= \eta_1 + \Phi_1(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3), \\ z_2 &= \eta_2 + \Phi_2(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3), \\ z_3 &= \eta_3 + \Phi_3(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3), \end{aligned} \right\} \quad \dots \dots \dots \dots \dots \quad (101.)$$

and

$$\left. \begin{aligned} \lambda_1 &= \varpi_1(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3), \\ \lambda_2 &= \varpi_2(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3), \\ \lambda_3 &= \varpi_3(t, \eta_1, \eta_2, \eta_3, \varpi_1, \varpi_2, \varpi_3); \end{aligned} \right\} \quad \dots \dots \dots \dots \dots \quad (102.)$$

we shall have, for such disturbed motion, the following rigorous equations, of the forms (94.) and (95.),

$$\left. \begin{aligned} \eta_1 &= \varphi_1(t, z_1, z_2, z_3, \lambda_1, \lambda_2, \lambda_3), \\ \eta_2 &= \varphi_2(t, z_1, z_2, z_3, \lambda_1, \lambda_2, \lambda_3), \\ \eta_3 &= \varphi_3(t, z_1, z_2, z_3, \lambda_1, \lambda_2, \lambda_3), \end{aligned} \right\} \quad \dots \dots \dots \dots \dots \quad (103.)$$

and

$$\left. \begin{array}{l} \pi_1 = \psi_1(t, x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3), \\ \pi_2 = \psi_2(t, x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3), \\ \pi_3 = \psi_3(t, x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3); \end{array} \right\} \quad \dots \quad (104).$$

and may call the quantities $x_1 x_2 x_3 \lambda_1 \lambda_2 \lambda_3$ the 6 *varying elements* of the motion. To determine these six varying elements, we may employ the six following rigorous equations in ordinary differentials of the first order, in which H_2 is supposed to have been expressed by (103.) and (104.) as a function of the elements and of the time :

$$\left. \begin{aligned} \frac{dx_1}{dt} &= \frac{\delta H_2}{\delta \lambda_1}, \quad \frac{dx_2}{dt} = \frac{\delta H_2}{\delta \lambda_2}, \quad \frac{dx_3}{dt} = \frac{\delta H_2}{\delta \lambda_3}, \\ \frac{d\lambda_1}{dt} &= -\frac{\delta H_2}{\delta x_1}, \quad \frac{d\lambda_2}{dt} = -\frac{\delta H_2}{\delta x_2}, \quad \frac{d\lambda_3}{dt} = -\frac{\delta H_2}{\delta x_3}; \end{aligned} \right\} \dots \quad (105.)$$

and the rigorous integrals of these 6 equations may be expressed in the following manner,

$$\left. \begin{aligned} \lambda_1 &= \frac{\delta E}{\delta x_1}, \quad \lambda_2 = \frac{\delta E}{\delta x_2}, \quad \lambda_3 = \frac{\delta E}{\delta x_3}, \\ p_1 &= -\frac{\delta E}{\delta e_1}, \quad p_2 = -\frac{\delta E}{\delta e_2}, \quad p_3 = -\frac{\delta E}{\delta e_3}, \end{aligned} \right\} \quad \dots \quad (106.)$$

the constants $e_1 e_2 e_3 p_1 p_2 p_3$ retaining their recent meanings, and being therefore the initial values of the elements $z_1 z_2 z_2 \lambda_1 \lambda_2 \lambda_3$; while the function E , which may be called the *function of elements*, because its form determines the laws of their variations, is the definite integral

$$E = \int_0^t \left(\lambda_1 \frac{\delta H_2}{\delta \lambda_1} + \lambda_2 \frac{\delta H_2}{\delta \lambda_2} + \lambda_3 \frac{\delta H_2}{\delta \lambda_3} - H_2 \right) dt, \quad \dots \quad (107.)$$

considered as depending on $z_1 z_2 z_3 e_1 e_2 e_3$ and t . The integrals of the equations (105.) may also be expressed in this other way,

$$\left. \begin{array}{l} z_1 = +\frac{\delta C}{\delta \lambda_1}, \quad z_2 = +\frac{\delta C}{\delta \lambda_2}, \quad z_3 = +\frac{\delta C}{\delta \lambda_3}, \\ e_1 = -\frac{\delta C}{\delta p_1}, \quad e_2 = -\frac{\delta C}{\delta p_2}, \quad e_3 = -\frac{\delta C}{\delta p_3}, \end{array} \right\} \quad \dots \quad (108)$$

C being the definite integral

$$C = - \int_0^t \left(z_1 \frac{\delta H_2}{\delta x_1} + z_2 \frac{\delta H_2}{\delta x_2} + z_3 \frac{\delta H_2}{\delta x_3} - H_2 \right) dt, \quad \dots \quad (109.)$$

regarded as a function of $\lambda_1 \lambda_2 \lambda_3 p_1 p_2 p_3$ and t : and it is easy to prove that each of these two *functions of elements*, C and E, must satisfy a partial differential equation of the first order, which can be previously assigned, and which may assist in discovering the forms of these two functions, and especially in improving an approximate expression for either. All these results for the motion of a single point, are analogous to the results already deduced in this Essay, for an attracting or repelling system.

Mathematical Example, suggested by the motion of Projectiles.

24. If the three marks of position $\eta_1 \eta_2 \eta_3$ of the moving point are the rectangular coordinates themselves, and if the function U has the form

$$U = -g\eta_3 - \frac{1}{2}\{\mu^2(\eta_1^2 + \eta_2^2) + \nu^2\eta_3^2\}, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (110.)$$

g, μ, ν , being constants; then the expression

$$H = \frac{1}{2}(\varpi_1^2 + \varpi_2^2 + \varpi_3^2) + g\eta_3 + \frac{1}{2}\{\mu^2(\eta_1^2 + \eta_2^2) + \nu^2\eta_3^2\}, \quad \dots \quad \dots \quad (111.)$$

is that which must be substituted in the general forms (83.), in order to form the 6 differential equations of motion of the first order, namely,

$$\left. \begin{aligned} \frac{d\eta_1}{dt} &= \varpi_1, & \frac{d\eta_2}{dt} &= \varpi_2, & \frac{d\eta_3}{dt} &= \varpi_3, \\ \frac{d\varpi_1}{dt} &= -\mu^2\eta_1, & \frac{d\varpi_2}{dt} &= -\mu^2\eta_2, & \frac{d\varpi_3}{dt} &= -g - \nu^2\eta_3. \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad (112.)$$

These differential equations have for their rigorous integrals the six following,

$$\left. \begin{aligned} \eta_1 &= e_1 \cos \mu t + \frac{p_1}{\mu} \sin \mu t, \\ \eta_2 &= e_2 \cos \mu t + \frac{p_2}{\mu} \sin \mu t, \\ \eta_3 &= e_3 \cos \nu t + \frac{p_3}{\nu} \sin \nu t - \frac{g}{\nu^2} \operatorname{vers} \nu t, \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (113.)$$

and

$$\left. \begin{aligned} \varpi_1 &= p_1 \cos \mu t - \mu e_1 \sin \mu t, \\ \varpi_2 &= p_2 \cos \mu t - \mu e_2 \sin \mu t, \\ \varpi_3 &= p_3 \cos \nu t - (\nu e_3 + \frac{g}{\nu}) \sin \nu t, \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (114.)$$

$e_1 e_2 e_3 p_1 p_2 p_3$ being still the initial values of $\eta_1 \eta_2 \eta_3 \varpi_1 \varpi_2 \varpi_3$.

Employing these rigorous integral equations to calculate the function S , that is, by (85.) and (110.) (111.), the definite integral

$$S = \int_0^t \left(\frac{\varpi_1^2 + \varpi_2^2 + \varpi_3^2}{2} + U \right) dt, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (115.)$$

we find

$$\left. \begin{aligned} \frac{1}{2}(\varpi_1^2 + \varpi_2^2 + \varpi_3^2) &= \frac{1}{4} \left\{ p_1^2 + p_2^2 + p_3^2 + \mu^2(e_1^2 + e_2^2) + \left(\nu e_3 + \frac{g}{\nu}\right)^2 \right\} \\ + \frac{1}{4} \{p_1^2 + p_2^2 - \mu^2(e_1^2 + e_2^2)\} \cos 2\mu t - \frac{1}{2}\mu(e_1 p_1 + e_2 p_2) \sin 2\mu t \\ + \frac{1}{4} \left\{ p_3^2 - \left(\nu e_3 + \frac{g}{\nu}\right)^2 \right\} \cos 2\nu t - \frac{1}{2}\left(\nu e_3 + \frac{g}{\nu}\right) p_3 \sin 2\nu t, \end{aligned} \right\} \quad (116.)$$

and

$$\left. \begin{aligned} U &= \frac{g^2}{2\nu^2} - \frac{1}{4} \left\{ p_1^2 + p_2^2 + p_3^2 + \mu^2(e_1^2 + e_2^2) + \left(\nu e_3 + \frac{g}{\nu}\right)^2 \right\} \\ + \frac{1}{4} \{p_1^2 + p_2^2 - \mu^2(e_1^2 + e_2^2)\} \cos 2\mu t - \frac{1}{2}\mu(e_1 p_1 + e_2 p_2) \sin 2\mu t \\ + \frac{1}{4} \left\{ p_3^2 - \left(\nu e_3 + \frac{g}{\nu}\right)^2 \right\} \cos 2\nu t - \frac{1}{2}\left(\nu e_3 + \frac{g}{\nu}\right) p_3 \sin 2\nu t; \end{aligned} \right\} \quad (117.)$$

and therefore,

$$S = \frac{g^2 t}{2 v^2} + \{p_1^2 + p_2^2 - \mu^2 (e_1^2 + e_2^2)\} \frac{\sin 2 \mu t}{4 \mu} - \frac{1}{2} (e_1 p_1 + e_2 p_2) \operatorname{vers} 2 \mu t + \left\{ p_3^2 - \left(v e_3 + \frac{g}{v}\right)^2 \right\} \frac{\sin 2 v t}{4 v} - \frac{1}{2} p_3 \left(e_3 + \frac{g}{v^2}\right) \operatorname{vers} 2 v t \quad (118.)$$

In order, however, to express this function S , as supposed by our general method, in terms of the final and initial coordinates and of the time, we must employ the analogous expressions for the constants $p_1 p_2 p_3$, deduced from the integrals (113.), namely, the following :

$$\left. \begin{aligned} p_1 &= \frac{\mu \eta_1 - \mu e_1 \cos \mu t}{\sin \mu t}, \\ p_2 &= \frac{\mu \eta_2 - \mu e_2 \cos \mu t}{\sin \mu t}, \\ p_3 &= \frac{v \eta_3 + \frac{g}{v} - \left(v e_3 + \frac{g}{v^2}\right) \cos v t}{\sin v t}; \end{aligned} \right\} \quad \dots \dots \dots \quad (119.)$$

and then we find

$$\left. \begin{aligned} S &= \frac{g^2 t}{2 v^2} + \frac{\mu}{2} \cdot \frac{(\eta_1 - e_1)^2 + (\eta_2 - e_2)^2}{\tan \mu t} + \frac{v}{2} \cdot \frac{(\eta_3 - e_3)^2}{\tan v t} \\ &\quad - \mu (\eta_1 e_1 + \eta_2 e_2) \tan \frac{\mu t}{2} - v (\eta_3 + \frac{g}{v^2}) (e_3 + \frac{g}{v^2}) \tan \frac{v t}{2}. \end{aligned} \right\} \quad \dots \dots \quad (120.)$$

This *principal function* S satisfies the following pair of partial differential equations of the first order, of the kind (86.),

$$\left. \begin{aligned} \frac{\delta S}{\delta t} + \frac{1}{2} \left\{ \left(\frac{\delta S}{\delta \eta_1}\right)^2 + \left(\frac{\delta S}{\delta \eta_2}\right)^2 + \left(\frac{\delta S}{\delta \eta_3}\right)^2 \right\} &= -g \eta_3 - \frac{\mu^2}{2} (\eta_1^2 + \eta_2^2) - \frac{v^2}{2} \eta_3^2, \\ \frac{\delta S}{\delta t} + \frac{1}{2} \left\{ \left(\frac{\delta S}{\delta e_1}\right)^2 + \left(\frac{\delta S}{\delta e_2}\right)^2 + \left(\frac{\delta S}{\delta e_3}\right)^2 \right\} &= -g e_3 - \frac{\mu^2}{2} (e_1^2 + e_2^2) - \frac{v^2}{2} e_3^2; \end{aligned} \right\} \quad (121.)$$

and if its form had been previously found, by the help of this pair, or in any other way, the integrals of the equations of motion might (by our general method) have been deduced from it, under the forms,

$$\left. \begin{aligned} \pi_1 &= \frac{\delta S}{\delta \eta_1} = \mu (\eta_1 - e_1) \cotan \mu t - \mu e_1 \tan \frac{\mu t}{2}, \\ \pi_2 &= \frac{\delta S}{\delta \eta_2} = \mu (\eta_2 - e_2) \cotan \mu t - \mu e_2 \tan \frac{\mu t}{2}, \\ \pi_3 &= \frac{\delta S}{\delta \eta_3} = v (\eta_3 - e_3) \cotan v t - \left(v e_3 + \frac{g}{v}\right) \tan \frac{v t}{2}, \end{aligned} \right\} \quad \dots \dots \quad (122.)$$

and

$$\left. \begin{aligned} p_1 &= -\frac{\delta S}{\delta e_1} = \mu (\eta_1 - e_1) \cotan \mu t + \mu \eta_1 \tan \frac{\mu t}{2}, \\ p_2 &= -\frac{\delta S}{\delta e_2} = \mu (\eta_2 - e_2) \cotan \mu t + \mu \eta_2 \tan \frac{\mu t}{2}, \\ p_3 &= -\frac{\delta S}{\delta e_3} = v (\eta_3 - e_3) \cotan v t + \left(v \eta_3 + \frac{g}{v}\right) \tan \frac{v t}{2}; \end{aligned} \right\} \quad \dots \quad (123.)$$

the last of these two sets of equations coinciding with the set (119.), or (113.), and conducting, when combined with the first set, (122.), to the other former set of integrals, (114.).

25. Suppose now, to illustrate the theory of perturbation, that the constants μ , ν are small, and that, after separating the expression (111.) for H into the two parts,

$$H_1 = \frac{1}{2}(\varpi_1^2 + \varpi_2^2 + \varpi_3^2) + g \eta_3, \quad \dots \quad (124.)$$

and

$$H_2 = \frac{1}{2}\{\mu^2(\eta_1^2 + \eta_2^2) + \nu^2\eta_3^2\}, \quad \dots \quad (125.)$$

we suppress at first the small part H_2 , and so form, by (88.), these other and simpler differential equations of a motion which we shall call *undisturbed*:

$$\left. \begin{aligned} \frac{d\eta_1}{dt} &= \varpi_1, & \frac{d\eta_2}{dt} &= \varpi_2, & \frac{d\eta_3}{dt} &= \varpi_3, \\ \frac{d\varpi_1}{dt} &= 0, & \frac{d\varpi_2}{dt} &= 0, & \frac{d\varpi_3}{dt} &= -g. \end{aligned} \right\} \quad \dots \quad (126.)$$

These new equations have for their rigorous integrals, of the forms (94.) and (95.),

$$\eta_1 = e_1 + p_1 t, \quad \eta_2 = e_2 + p_2 t, \quad \eta_3 = e_3 + p_3 t - \frac{1}{2}g t^2, \quad \dots \quad (127.)$$

and

$$\varpi_1 = p_1, \quad \varpi_2 = p_2, \quad \varpi_3 = p_3 - g t; \quad \dots \quad (128.)$$

and the *principal function* S_1 of the same undisturbed motion is, by (89.),

$$\left. \begin{aligned} S_1 &= \int_0^t \left(\frac{\varpi_1^2 + \varpi_2^2 + \varpi_3^2}{2} - g \eta_3 \right) dt \\ &= \int_0^t \left(\frac{p_1^2 + p_2^2 + p_3^2}{2} - g e_3 - 2 g p_3 t + g^2 t^2 \right) dt \\ &= \left(\frac{p_1^2 + p_2^2 + p_3^2}{2} - g e_3 \right) t - g p_3 t^2 + \frac{1}{3}g^2 t^3, \end{aligned} \right\} \quad \dots \quad (129.)$$

or finally, by (127.),

$$S_1 = \frac{(\eta_1 - e_1)^2 + (\eta_2 - e_2)^2 + (\eta_3 - e_3)^2}{2t} - \frac{1}{2}g t(\eta_3 + e_3) - \frac{1}{24}g^2 t^3. \quad \dots \quad (130.)$$

This function satisfies, as it ought, the following pair of partial differential equations,

$$\left. \begin{aligned} \frac{\delta S_1}{\delta t} + \frac{1}{2} \left\{ \left(\frac{\delta S_1}{\delta \eta_1} \right)^2 + \left(\frac{\delta S_1}{\delta \eta_2} \right)^2 + \left(\frac{\delta S_1}{\delta \eta_3} \right)^2 \right\} &= -g \eta_3, \\ \frac{\delta S_1}{\delta t} + \frac{1}{2} \left\{ \left(\frac{\delta S_1}{\delta e_1} \right)^2 + \left(\frac{\delta S_1}{\delta e_2} \right)^2 + \left(\frac{\delta S_1}{\delta e_3} \right)^2 \right\} &= -g e_3; \end{aligned} \right\} \quad \dots \quad (131.)$$

And if by the help of this pair, or in any other way, the form (130.) of this *principal function* S_1 had been found, the integral equations (127.) and (128.) might have been deduced from it, by our general method, as follows:

$$\left. \begin{aligned} \varpi_1 &= \frac{\delta S_1}{\delta \eta_1} = \frac{\eta_1 - e_1}{t}, \\ \varpi_2 &= \frac{\delta S_1}{\delta \eta_2} = \frac{\eta_2 - e_2}{t}, \\ \varpi_3 &= \frac{\delta S_1}{\delta \eta_3} = \frac{\eta_3 - e_3}{t} - \frac{1}{2}g t, \end{aligned} \right\} \quad \dots \quad (132.)$$

and

$$\left. \begin{aligned} p_1 &= -\frac{\delta S_1}{\delta e_1} = \frac{\eta_1 - e_1}{t}, \\ p_2 &= -\frac{\delta S_1}{\delta e_2} = \frac{\eta_2 - e_2}{t}, \\ p_3 &= -\frac{\delta S_1}{\delta e_3} = \frac{\eta_3 - e_3}{t} + \frac{1}{2} g t; \end{aligned} \right\} \quad \dots \quad (133.)$$

the latter of these two sets coinciding with (127.), and the former set conducting to (128.).

26. Returning now from this simpler motion to the more complex motion first mentioned, and denoting by S_2 the *disturbing part* or function which must be added to S_1 in order to make up the whole principal function S of that more complex motion; we have, by applying our general method, the following rigorous expression for this disturbing function,

$$S_2 = - \int_0^t H_2 dt + \int_0^t \frac{1}{2} \left\{ \left(\frac{\delta S_2}{\delta \eta_1} \right)^2 + \left(\frac{\delta S_2}{\delta \eta_2} \right)^2 + \left(\frac{\delta S_2}{\delta \eta_3} \right)^2 \right\} dt, \quad (134.)$$

in which we may, approximately, neglect the second definite integral, and calculate the first by the help of the equations of undisturbed motion. In this manner we find, approximately, by (125.) (127.),

$$-H_2 = -\frac{\mu^2}{2} \left\{ (e_1 + p_1 t)^2 + (e_2 + p_2 t)^2 \right\} - \frac{\nu^2}{2} (e_3 + p_3 t - \frac{1}{2} g t^2)^2, \quad (135.)$$

and therefore, by integration,

$$\left. \begin{aligned} S_2 &= -\frac{1}{2} \{ \mu^2 (e_1^2 + e_2^2) + \nu^2 e_3^2 \} t - \frac{1}{2} \{ \mu^2 (e_1 p_1 + e_2 p_2) + \nu^2 e_3 p_3 \} t^2 \\ &\quad - \frac{1}{6} \{ \mu^2 (p_1^2 + p_2^2) + \nu^2 (p_3^2 - g e_3) \} t^3 + \frac{1}{8} \nu^2 g p_3 t^4 - \frac{1}{40} \nu^2 g^2 t^5, \end{aligned} \right\} \quad (136.)$$

or, by (133.),

$$\left. \begin{aligned} S_2 &= -\frac{\mu^2 t}{6} (\eta_1^2 + e_1 \eta_1 + e_1^2 + \eta_2^2 + e_2 \eta_2 + e_2^2) \\ &\quad - \frac{\nu^2 t}{6} \left\{ \eta_3^2 + e_3 \eta_3 + e_3^2 + \frac{1}{4} g (\eta_3 + e_3) t^2 + \frac{1}{40} g^2 t^4 \right\} : \end{aligned} \right\} \quad \dots \quad (137.)$$

the error being of the fourth order, with respect to the small quantities μ , ν . And neglecting this small error, we can deduce, by our general method, approximate forms for the integrals of the equations of disturbed motion, from the corrected function $S_1 + S_2$, as follows:

$$\left. \begin{aligned} \varpi_1 &= \frac{\delta S_1}{\delta \eta_1} + \frac{\delta S_2}{\delta \eta_1} = \frac{\eta_1 - e_1}{t} - \frac{\mu^2 t}{3} \left(\eta_1 + \frac{1}{2} e_1 \right), \\ \varpi_2 &= \frac{\delta S_1}{\delta \eta_2} + \frac{\delta S_2}{\delta \eta_2} = \frac{\eta_2 - e_2}{t} - \frac{\mu^2 t}{3} \left(\eta_2 + \frac{1}{2} e_2 \right), \\ \varpi_3 &= \frac{\delta S_1}{\delta \eta_3} + \frac{\delta S_2}{\delta \eta_3} = \frac{\eta_3 - e_3}{t} - \frac{1}{2} g t - \frac{\nu^2 t}{3} \left(\eta_3 + \frac{1}{2} e_3 + \frac{1}{8} g t^2 \right); \end{aligned} \right\} \quad \dots \quad (138.)$$

and

$$\left. \begin{aligned} p_1 &= -\frac{\delta S_1}{\delta e_1} - \frac{\delta S_2}{\delta e_1} = \frac{\eta_1 - e_1}{t} + \frac{\mu^2 t}{3} \left(e_1 + \frac{1}{2} \eta_1 \right), \\ p_2 &= -\frac{\delta S_1}{\delta e_2} - \frac{\delta S_2}{\delta e_2} = \frac{\eta_2 - e_2}{t} + \frac{\mu^2 t}{3} \left(e_2 + \frac{1}{2} \eta_2 \right), \\ p_3 &= -\frac{\delta S_1}{\delta e_3} - \frac{\delta S_2}{\delta e_3} = \frac{\eta_3 - e_3}{t} + \frac{1}{2} g t + \frac{\nu^2 t}{3} \left(e_3 + \frac{1}{2} \eta_3 + \frac{1}{8} g t^2 \right); \end{aligned} \right\} \quad (139.)$$

or, in the same order of approximation,

$$\left. \begin{aligned} \eta_1 &= e_1 + p_1 t - \frac{1}{2} \mu^2 t^2 \left(e_1 + \frac{1}{3} p_1 t \right), \\ \eta_2 &= e_2 + p_2 t - \frac{1}{2} \mu^2 t^2 \left(e_2 + \frac{1}{3} p_2 t \right), \\ \eta_3 &= e_3 + p_3 t - \frac{1}{2} g t^2 - \frac{1}{2} \nu^2 t^2 \left(e_3 + \frac{1}{3} p_3 t - \frac{1}{12} g t^2 \right), \end{aligned} \right\} \quad \dots \quad (140.)$$

and

$$\left. \begin{aligned} \varpi_1 &= p_1 - \mu^2 t \left(e_1 + \frac{1}{2} p_1 t \right), \\ \varpi_2 &= p_2 - \mu^2 t \left(e_2 + \frac{1}{2} p_2 t \right), \\ \varpi_3 &= p_3 - g t - \nu^2 t \left(e_3 + \frac{1}{2} p_3 t - \frac{1}{6} g t^2 \right). \end{aligned} \right\} \quad \dots \quad (141.)$$

Accordingly, if we develope the rigorous integrals of disturbed motion, (113.) and (114.), as far as the squares (inclusive) of the small quantities μ and ν , we are conducted to these approximate integrals; and if we develope the rigorous expression (120.) for the principal function of such motion, to the same degree of accuracy, we obtain the sum of the two expressions (130.) and (137.).

27. To illustrate still further, in the present example, our general method of successive approximation, let S_3 denote the small unknown correction of the approximate expression (137.), so that we shall now have, rigorously, for the present disturbed motion,

$$S = S_1 + S_2 + S_3, \quad \dots \quad (142.)$$

S_1 and S_2 being here determined rigorously by (130.) and (137.). Then, substituting $S_1 + S_2$ for S_1 in the general transformation (87.), we find, rigorously, in the present question,

$$\left. \begin{aligned} S_3 &= - \int_0^t \frac{1}{2} \left\{ \left(\frac{\delta S_2}{\delta \eta_1} \right)^2 + \left(\frac{\delta S_2}{\delta \eta_2} \right)^2 + \left(\frac{\delta S_2}{\delta \eta_3} \right)^2 \right\} dt \\ &\quad + \int_0^t \frac{1}{2} \left\{ \left(\frac{\delta S_3}{\delta \eta_1} \right)^2 + \left(\frac{\delta S_3}{\delta \eta_2} \right)^2 + \left(\frac{\delta S_3}{\delta \eta_3} \right)^2 \right\} dt; \end{aligned} \right\} \quad \dots \quad (143.)$$

and if we neglect only terms of the eighth and higher dimensions with respect to the small quantities μ , ν , we may confine ourselves to the first of these two definite integrals, and may employ, in calculating it, the approximate expressions (140.) for

the coordinates of disturbed motion. In this manner we obtain the very approximate expression,

$$\left. \begin{aligned} S_3 &= -\frac{\mu^4}{18} \int_0^t t^2 \left\{ \left(\eta_1 + \frac{1}{2} e_1 \right)^2 + \left(\eta_2 + \frac{1}{2} e_2 \right)^2 \right\} dt \\ &\quad - \frac{\nu^4}{18} \int_0^t t^2 \left(\eta_3 + \frac{1}{2} e_3 + \frac{1}{8} g t^2 \right)^2 dt \\ &= -\frac{\mu^4 t^3}{360} (4 \eta_1^2 + 7 \eta_1 e_1 + 4 e_1^2 + 4 \eta_2^2 + 7 \eta_2 e_2 + 4 e_2^2) \\ &\quad - \frac{\nu^4 t^3}{360} (4 \eta_3^2 + 7 \eta_3 e_3 + 4 e_3^2) - \frac{\nu^4 g t^5}{240} (\eta_3 + e_3) - \frac{17 \nu^4 g^2 t^7}{40320} \\ &\quad - \frac{\mu^6 t^5}{945} \left(\eta_1^2 + \frac{31}{16} \eta_1 e_1 + e_1^2 + \eta_2^2 + \frac{31}{16} \eta_2 e_2 + e_2^2 \right) \\ &\quad - \frac{\nu^6 t^5}{945} \left(\eta_3^2 + \frac{31}{16} \eta_3 e_3 + e_3^2 \right) - \frac{17 \nu^6 g t^7 (\eta_3 + e_3)}{40320} - \frac{31 \nu^6 g^2 t^9}{725760}; \end{aligned} \right\} \quad (144.)$$

which is accordingly the sum of the terms of the fourth and sixth dimensions in the development of the rigorous expression (120.), and gives, by our general method, correspondingly approximate expressions for the integrals of disturbed motion, under the forms

$$\left. \begin{aligned} \varpi_1 &= \frac{\delta S_1}{\delta \eta_1} + \frac{\delta S_2}{\delta \eta_1} + \frac{\delta S_3}{\delta \eta_1}, \\ \varpi_2 &= \frac{\delta S_1}{\delta \eta_2} + \frac{\delta S_2}{\delta \eta_2} + \frac{\delta S_3}{\delta \eta_2}, \\ \varpi_3 &= \frac{\delta S_1}{\delta \eta_3} + \frac{\delta S_2}{\delta \eta_3} + \frac{\delta S_3}{\delta \eta_3}, \end{aligned} \right\} \quad \dots \quad (145.)$$

and

$$\left. \begin{aligned} p_1 &= -\frac{\delta S_1}{\delta e_1} - \frac{\delta S_2}{\delta e_1} - \frac{\delta S_3}{\delta e_1}, \\ p_2 &= -\frac{\delta S_1}{\delta e_2} - \frac{\delta S_2}{\delta e_2} - \frac{\delta S_3}{\delta e_2}, \\ p_3 &= -\frac{\delta S_1}{\delta e_3} - \frac{\delta S_2}{\delta e_3} - \frac{\delta S_3}{\delta e_3}. \end{aligned} \right\} \quad \dots \quad (146.)$$

28. To illustrate by the same example the theory of gradually varying elements, let us establish the following definitions, for the present disturbed motion,

$$\left. \begin{aligned} z_1 &= \eta_1 - \varpi_1 t, \quad z_2 = \eta_2 - \varpi_2 t, \quad z_3 = \eta_3 - \varpi_3 t - \frac{1}{2} g t^2, \\ \lambda_1 &= \varpi_1, \quad \lambda_2 = \varpi_2, \quad \lambda_3 = \varpi_3 + g t, \end{aligned} \right\} \quad \dots \quad (147.)$$

and let us call these six quantities $z_1 z_2 z_3 \lambda_1 \lambda_2 \lambda_3$ the *varying elements* of that motion, by analogy to the six constant quantities $e_1 e_2 e_3 p_1 p_2 p_3$, which may, for the undisturbed motion, be represented in a similar way, namely, by (127.) and (128.),

$$\left. \begin{aligned} e_1 &= \eta_1 - \varpi_1 t, & e_2 &= \eta_2 - \varpi_2 t, & e_3 &= \eta_3 - \varpi_3 t - \frac{1}{2} g t^2, \\ p_1 &= \varpi_1, & p_2 &= \varpi_2, & p_3 &= \varpi_3 + g t. \end{aligned} \right\} \dots \quad (148.)$$

We shall then have rigorously, for the six disturbed variables $\eta_1 \eta_2 \eta_3 \varpi_1 \varpi_2 \varpi_3$, expressions of the same forms as in the integrals (127.) and (128.) of undisturbed motion, but with variable instead of constant elements, namely, the following:

$$\left. \begin{aligned} \eta_1 &= z_1 + \lambda_1 t, & \eta_2 &= z_2 + \lambda_2 t, & \eta_3 &= z_3 + \lambda_3 t - \frac{1}{2} g t^2, \\ \varpi_1 &= \lambda_1, & \varpi_2 &= \lambda_2, & \varpi_3 &= \lambda_3 - g t; \end{aligned} \right\} \dots \quad (149.)$$

and the rigorous determination of the six varying elements $z_1 z_2 z_3 \lambda_1 \lambda_2 \lambda_3$, as functions of the time and of their own initial values $e_1 e_2 e_3 p_1 p_2 p_3$, depends on the integration of the 6 following equations, in ordinary differentials of the first order, of the forms (105.):

$$\left. \begin{aligned} \frac{dx_1}{dt} &= \frac{\delta H_2}{\delta \lambda_1} = \mu^2 t (z_1 + \lambda_1 t), \\ \frac{dx_2}{dt} &= \frac{\delta H_2}{\delta \lambda_2} = \mu^2 t (z_2 + \lambda_2 t), \\ \frac{dx_3}{dt} &= \frac{\delta H_2}{\delta \lambda_3} = \nu^2 t \left(z_3 + \lambda_3 t - \frac{1}{2} g t^2 \right), \end{aligned} \right\} \dots \quad (150.)$$

and

$$\left. \begin{aligned} \frac{d\lambda_1}{dt} &= - \frac{\delta H_2}{\delta z_1} = - \mu^2 (z_1 + \lambda_1 t), \\ \frac{d\lambda_2}{dt} &= - \frac{\delta H_2}{\delta z_2} = - \mu^2 (z_2 + \lambda_2 t), \\ \frac{d\lambda_3}{dt} &= - \frac{\delta H_2}{\delta z_3} = - \nu^2 \left(z_3 + \lambda_3 t - \frac{1}{2} g t^2 \right), \end{aligned} \right\} \dots \quad (151.)$$

H_2 being here the expression

$$H_2 = \frac{\mu^2}{2} \{ (z_1 + \lambda_1 t)^2 + (z_2 + \lambda_2 t)^2 \} + \frac{\nu^2}{2} \left(z_3 + \lambda_3 t - \frac{1}{2} g t^2 \right)^2, \quad (152.)$$

which is obtained from (125.) by substituting for the disturbed coordinates $\eta_1 \eta_2 \eta_3$ their values (149.), as functions of the varying elements and of the time. It is not difficult to integrate rigorously this system of equations (150.) and (151.); and we shall soon have occasion to state their complete and accurate integrals: but we shall continue for a while to treat these rigorous integrals as unknown, that we may take this opportunity to exemplify our general method of indefinite approximation, for all such dynamical questions, founded on the properties of the *functions of elements* C and E. Of these two functions either may be employed, and we shall use here the function C.

29. This function, by (109.) and (152.), may rigorously be expressed as follows:

$$C = \frac{\mu^2}{2} \int_0^t (\lambda_1^2 t^2 - z_1^2 + \lambda_2^2 t^2 - z_2^2) dt + \frac{v^2}{2} \int_0^t \left\{ (\lambda_3 t - \frac{1}{2} g t^2)^2 - z_3^2 \right\} dt; \quad \dots \dots \dots \dots \quad (153.)$$

and has therefore the following for a first approximate value, obtained by treating the elements $z_1 z_2 z_3 \lambda_1 \lambda_2 \lambda_3$ as constant and equal to their initial values $e_1 e_2 e_3 p_1 p_2 p_3$,

$$C = -\frac{t}{2} \left\{ \mu^2 (e_1^2 + e_2^2) + v^2 e_3^2 \right\} + \frac{t^3}{6} \left\{ \mu^2 (p_1^2 + p_2^2) + v^2 p_3^2 \right\} - \frac{t^4}{8} v^2 g p_3 + \frac{t^5}{40} v^2 g^2. \quad \dots \dots \dots \quad (154.)$$

In like manner we have, as first approximations, of the kind expressed by the general formula (Z¹), the following results deduced from the equations (151.),

$$\left. \begin{aligned} \lambda_1 &= p_1 - \mu^2 \left(e_1 t + \frac{1}{2} p_1 t^2 \right), \\ \lambda_2 &= p_2 - \mu^2 \left(e_2 t + \frac{1}{2} p_2 t^2 \right), \\ \lambda_3 &= p_3 - v^2 \left(e_3 t + \frac{1}{2} p_3 t^2 - \frac{1}{6} g t^3 \right), \end{aligned} \right\} \dots \dots \dots \quad (155.)$$

and therefore, as approximations of the same kind,

$$\left. \begin{aligned} e_1 &= -\frac{1}{2} p_1 t - \frac{\lambda_1 - p_1}{\mu^2 t}, \\ e_2 &= -\frac{1}{2} p_2 t - \frac{\lambda_2 - p_2}{\mu^2 t}, \\ e_3 &= -\frac{1}{2} p_3 t + \frac{1}{6} g t^2 - \frac{\lambda_3 - p_3}{v^2 t}. \end{aligned} \right\} \dots \dots \dots \quad (156.)$$

Substituting these values for the initial constants $e_1 e_2 e_3$ in the approximate value (154.) for the function of elements C, we obtain the following approximate expression C₁ for that function, of the form supposed by our theory :

$$\left. \begin{aligned} C_1 &= -\frac{1}{2} t \left\{ \frac{(\lambda_1 - p_1)^2 + (\lambda_2 - p_2)^2 + (\lambda_3 - p_3)^2}{\mu^2} + \frac{v^2}{2} \right\} \\ &\quad - \frac{t}{2} \left\{ (\lambda_1 - p_1) p_1 + (\lambda_2 - p_2) p_2 + (\lambda_3 - p_3) (p_3 - \frac{1}{3} g t) \right\} \\ &\quad + \frac{t^3}{24} \left\{ \mu^2 (p_1^2 + p_2^2) + v^2 p_3^2 \right\} - \frac{t^4}{24} v^2 g p_3 + \frac{t^5}{90} v^2 g^2. \end{aligned} \right\} \quad (157.)$$

The rigorous function C must satisfy, in the present question, by the principles of the eighteenth number, the partial differential equation,

$$\frac{\partial C}{\partial t} = \frac{\mu^2}{2} \left\{ \left(\frac{\partial C}{\partial \lambda_1} + \lambda_1 t \right)^2 + \left(\frac{\partial C}{\partial \lambda_2} + \lambda_2 t \right)^2 \right\} + \frac{v^2}{2} \left(\frac{\partial C}{\partial \lambda_3} + \lambda_3 t - \frac{1}{2} g t^2 \right)^2; \quad (158.)$$

and if it be put under the form (U¹.),

$$C = C_1 + C_2,$$

C₁ being a first approximation, supposed to vanish with the time, then the correction C₂ must satisfy rigorously the condition

$$C_2 = \left. \int_0^t \left\{ -\frac{\delta C_1}{\delta t} + \frac{\mu^2}{2} \left(\frac{\delta C_1}{\delta \lambda_1} + \lambda_1 t \right)^2 + \frac{\mu^2}{2} \left(\frac{\delta C_1}{\delta \lambda_2} + \lambda_2 t \right)^2 + \frac{v^2}{2} \left(\frac{\delta C_1}{\delta \lambda_3} + \lambda_3 t - \frac{1}{2} g t^2 \right)^2 \right\} dt \right\} (159.)$$

$$- \frac{1}{2} \int_0^t \left\{ \mu^2 \left(\frac{\delta C_2}{\delta \lambda_1} \right)^2 + \mu^2 \left(\frac{\delta C_2}{\delta \lambda_2} \right)^2 + v^2 \left(\frac{\delta C_2}{\delta \lambda_3} \right)^2 \right\} dt.$$

In passing to a second approximation we may neglect the second definite integral, and may calculate the first by the help of the approximate equations (155.); which give, in this manner,

$$C_2 = - \left. \int_0^t \left\{ (\lambda_1 - p_1)^2 + (\lambda_2 - p_2)^2 + (\lambda_3 - p_3)^2 \right\} dt \right. \\ + \frac{\mu^2}{2} \int_0^t \left\{ \lambda_1 (\lambda_1 - p_1) + \lambda_2 (\lambda_2 - p_2) \right\} dt \\ + \frac{v^2}{2} \int_0^t \left(\lambda_3 - \frac{v^2}{3} g t \right) (\lambda_3 - p_3) t^2 dt \\ = - \frac{t}{3} \{ (\lambda_1 - p_1)^2 + (\lambda_2 - p_2)^2 + (\lambda_3 - p_3)^2 \} \\ + \frac{t^3}{24} \{ \mu^2 p_1 (\lambda_1 - p_1) + \mu^2 p_2 (\lambda_2 - p_2) + v^2 p_3 (\lambda_3 - p_3) \} \\ - \frac{t^4}{45} v^2 g (\lambda_3 - p_3) + \frac{t^5}{240} (\mu^4 p_1^2 + \mu^4 p_2^2 + v^4 p_3^2) \\ - \frac{t^6}{240} v^4 g p_3 + \frac{t^7}{945} v^4 g^2. \quad \right\} . \quad (160.)$$

We might improve this second approximation in like manner, by calculating a new definite integral C_3 , with the help of the following more approximate forms for the relations between the varying elements $\lambda_1 \lambda_2 \lambda_3$ and the initial constants, deduced by our general method :

$$e_1 = - \frac{\delta C_1}{\delta p_1} - \frac{\delta C_2}{\delta p_1} = - \frac{\lambda_1 - p_1}{\mu^2 t} \left(1 + \frac{\mu^2 t^2}{6} + \frac{\mu^4 t^4}{24} \right) - \frac{t p_1}{2} \left(1 + \frac{\mu^2 t^2}{12} + \frac{\mu^4 t^4}{60} \right),$$

$$e_2 = - \frac{\delta C_1}{\delta p_2} - \frac{\delta C_2}{\delta p_2} = - \frac{\lambda_2 - p_2}{\mu^2 t} \left(1 + \frac{\mu^2 t^2}{6} + \frac{\mu^4 t^4}{24} \right) - \frac{t p_2}{2} \left(1 + \frac{\mu^2 t^2}{12} + \frac{\mu^4 t^4}{60} \right),$$

$$e_3 = - \frac{\delta C_1}{\delta p_3} - \frac{\delta C_2}{\delta p_3} = - \frac{\lambda_3 - p_3}{v^2 t} \left(1 + \frac{v^2 t^2}{6} + \frac{v^4 t^4}{24} \right) - \frac{t p_3}{2} \left(1 + \frac{v^2 t^2}{12} + \frac{v^4 t^4}{60} \right) \\ + \frac{g t^2}{6} \left(1 + \frac{7 v^2 t^2}{60} + \frac{v^4 t^4}{40} \right); \quad \right\} . \quad (161.)$$

in which we can only depend on the terms as far as the second order, but which acquire a correctness of the fourth order when cleared of the small divisors, and give then

$$\lambda_1 = p_1 - \mu^2 t (e_1 + \frac{1}{2} p_1 t) + \frac{1}{6} \mu^4 t^3 (e_1 + \frac{1}{4} p_1 t),$$

$$\lambda_2 = p_2 - \mu^2 t (e_2 + \frac{1}{2} p_2 t) + \frac{1}{6} \mu^4 t^3 (e_2 + \frac{1}{4} p_2 t),$$

$$\lambda_3 = p_3 - v^2 t (e_3 + \frac{1}{2} p_3 t - \frac{1}{6} g t^2) + \frac{1}{6} v^4 t^3 (e_3 + \frac{1}{4} p_3 t - \frac{1}{20} g t^2). \quad \right\} . \quad (162.)$$

But a little attention to the nature of this process shows that all the successive corrections to which it conducts can be only rational and integer and homogeneous functions, of the second dimension, of the quantities $\lambda_1 \lambda_2 \lambda_3 p_1 p_2 p_3 g$, and that they may all be put under the following form, which is therefore the form of their sum, or of the whole sought function C;

$$\left. \begin{aligned} C = & \mu^{-2} a_\mu (\lambda_1 - p_1)^2 + b_\mu p_1 (\lambda_1 - p_1) + \mu^2 c_\mu p_1^2 \\ & + \mu^{-2} a_\mu (\lambda_2 - p_2)^2 + b_\mu p_2 (\lambda_2 - p_2) + \mu^2 c_\mu p_2^2 \\ & + \nu^{-2} a_\nu (\lambda_3 - p_3)^2 + b_\nu p_3 (\lambda_3 - p_3) + \nu^2 c_\nu p_3^2 \\ & + f_\nu g (\lambda_3 - p_3) + \nu^2 h_\nu g p_3 + \nu^2 i_\nu g^2, \end{aligned} \right\} \quad \dots \quad (163.)$$

the coefficients $a_\mu a_\nu$ &c. being functions of the small quantities μ, ν , and also of the time, of which it remains to discover the forms. Denoting therefore their differentials, taken with respect to the time, as follows,

$$d a_\mu = a'_\mu d t, \quad d a_\nu = a'_\nu d t, \quad \text{&c.}, \quad \dots \quad (164.)$$

and substituting the expression (163.) in the rigorous partial differential equation (158.), we are conducted to the six following equations in ordinary differentials of the first order :

$$\left. \begin{aligned} 2 a'_\nu &= (2 a_\nu + \nu^2 t)^2; \quad b'_\nu = (2 a_\nu + \nu^2 t) (b_\nu + t); \quad c'_\nu = \frac{1}{2} (b_\nu + t)^2; \\ f'_\nu &= (2 a_\nu + \nu^2 t) (f_\nu - \frac{1}{2} t^2); \quad h'_\nu = (b_\nu + t) (f_\nu - \frac{1}{2} t^2); \quad i'_\nu = \frac{1}{2} (f_\nu - \frac{1}{2} t^2)^2; \end{aligned} \right\} \quad (165.)$$

along with the 6 following conditions, to determine the 6 arbitrary constants introduced by integration,

$$a_0 = -\frac{1}{2} t; \quad b_0 = -\frac{t}{2}; \quad f_0 = \frac{t^2}{6}; \quad c_0 = \frac{t^3}{24}; \quad h_0 = -\frac{t^4}{24}; \quad i_0 = \frac{t^5}{90}. \quad \dots \quad (166.)$$

In this manner we find, without difficulty, observing that $a_\mu b_\mu c_\mu$ may be formed from a, b, c , by changing ν to μ ,

$$\left. \begin{aligned} a_\nu &= -\frac{1}{2} \nu^2 t - \frac{1}{2} \nu \cotan \nu t, & a_\mu &= -\frac{1}{2} \mu^2 t - \frac{1}{2} \mu \cotan \mu t, \\ b_\nu &= -t + \frac{1}{\nu} \tan \frac{\nu t}{2}, & b_\mu &= -t + \frac{1}{\mu} \tan \frac{\mu t}{2}, \\ c_\nu &= -\frac{t}{2 \nu^2} + \frac{1}{\nu^3} \tan \frac{\nu t}{2}, & c_\mu &= -\frac{t}{2 \mu^2} + \frac{1}{\mu^3} \tan \frac{\mu t}{2}, \\ f_\nu &= \frac{t}{2} t^2 - \frac{1}{\nu^5} + \frac{t}{\nu} \cotan \nu t, \\ h_\nu &= \frac{t^3}{2 \nu^3} - \frac{t}{\nu^3} \tan \frac{\nu t}{2}, \\ i_\nu &= \frac{t}{2 \nu^4} - \frac{t^3}{6 \nu^2} - \frac{t^2}{2 \nu^3} \cotan \nu t. \end{aligned} \right\} \quad \dots \quad (167.)$$

The form of the function C is therefore entirely known, and we have for this *function of elements* the following rigorous expression,

$$\left. \begin{aligned}
 C = & -\frac{(\lambda_1 - p_1)^2 + (\lambda_2 - p_2)^2 + (\lambda_3 - p_3)^2}{2\mu \tan \mu t} - \frac{(\lambda_3 - p_3)^2}{2\nu \tan \nu t} \\
 & - \frac{t}{2} \{(\lambda_1 - p_1)^2 + (\lambda_2 - p_2)^2 + (\lambda_3 - p_3)^2\} \\
 & - t \{p_1(\lambda_1 - p_1) + p_2(\lambda_2 - p_2) + p_3(\lambda_3 - p_3)\} \\
 & + \frac{1}{\mu} \{p_1(\lambda_1 - p_1) + p_2(\lambda_2 - p_2)\} \tan \frac{\mu t}{2} + \frac{1}{\nu} p_3(\lambda_3 - p_3) \tan \frac{\nu t}{2} \\
 & - \frac{t}{2} (p_1^2 + p_2^2 + p_3^2) + \frac{1}{\mu} (p_1^2 + p_2^2) \tan \frac{\mu t}{2} + \frac{1}{\nu} p_3^2 \tan \frac{\nu t}{2} \\
 & + \left(\frac{t^2}{2} - \frac{1}{\nu^2} + \frac{t}{\nu} \cotan \nu t \right) g(\lambda_3 - p_3) + \left(\frac{t^2}{2} - \frac{t}{\nu} \tan \frac{\nu t}{2} \right) g p_3 \\
 & + \left(\frac{t}{2\nu^2} - \frac{t^3}{6} - \frac{t^2}{2\nu} \cotan \nu t \right) g^2,
 \end{aligned} \right\} \quad (168.)$$

which may be variously transformed, and gives by our general method the following systems of rigorous integrals of the differential equations of varying elements, (150.), (151.):

$$\left. \begin{aligned}
 e_1 = -\frac{\delta C}{\delta p_1} = & -\frac{\lambda_1 - p_1}{\mu \sin \mu t} - \frac{p_1}{\mu} \tan \frac{\mu t}{2}, \\
 e_2 = -\frac{\delta C}{\delta p_2} = & -\frac{\lambda_2 - p_2}{\mu \sin \mu t} - \frac{p_2}{\mu} \tan \frac{\mu t}{2}, \\
 e_3 = -\frac{\delta C}{\delta p_3} = & -\frac{\lambda_3 - p_3}{\nu \sin \nu t} - \frac{p_3}{\nu} \tan \frac{\nu t}{2} + \frac{g}{\nu} \left(\frac{t}{\sin \nu t} - \frac{1}{\nu} \right),
 \end{aligned} \right\}. \quad (169.)$$

and

$$\left. \begin{aligned}
 z_1 = \frac{\delta C}{\delta \lambda_1} = & -(\lambda_1 - p_1) \left(t + \frac{1}{\mu} \cotan \mu t \right) + p_1 \left(-t + \frac{1}{\mu} \tan \frac{\mu t}{2} \right), \\
 z_2 = \frac{\delta C}{\delta \lambda_2} = & -(\lambda_2 - p_2) \left(t + \frac{1}{\mu} \cotan \mu t \right) + p_2 \left(-t + \frac{1}{\mu} \tan \frac{\mu t}{2} \right), \\
 z_3 = \frac{\delta C}{\delta \lambda_3} = & -(\lambda_3 - p_3) \left(t + \frac{1}{\nu} \cotan \nu t \right) + p_3 \left(-t + \frac{1}{\nu} \tan \frac{\nu t}{2} \right) \\
 & + g \left(\frac{t^2}{2} - \frac{1}{\nu^2} + \frac{t}{\nu} \cotan \nu t \right);
 \end{aligned} \right\}. \quad (170.)$$

that is,

$$\left. \begin{aligned}
 \lambda_1 = p_1 \cos \mu t - e_1 \mu \sin \mu t, \\
 \lambda_2 = p_2 \cos \mu t - e_2 \mu \sin \mu t, \\
 \lambda_3 = p_3 \cos \nu t - e_3 \nu \sin \nu t + g \left(t - \frac{1}{\nu} \sin \nu t \right),
 \end{aligned} \right\}. \quad \dots \quad (171.)$$

and

$$\left. \begin{aligned}
 z_1 = e_1 (\cos \mu t + \mu t \sin \mu t) + p_1 \left(\frac{1}{\mu} \sin \mu t - t \cos \mu t \right), \\
 z_2 = e_2 (\cos \mu t + \mu t \sin \mu t) + p_2 \left(\frac{1}{\mu} \sin \mu t - t \cos \mu t \right), \\
 z_3 = e_3 (\cos \nu t + \nu t \sin \nu t) + p_3 \left(\frac{1}{\nu} \sin \nu t - t \cos \nu t \right) \\
 - g \left(\frac{\operatorname{vers} \nu t}{\nu^2} - \frac{t}{\nu} \sin \nu t + \frac{t^2}{2} \right).
 \end{aligned} \right\}. \quad (172.)$$

Accordingly, these rigorous expressions for the 6 varying elements, in the present dynamical question, agree with the results obtained by the ordinary methods of integration from the 6 ordinary differential equations (150.) and (151.), and with those obtained by elimination from the equations (113.) (114.) (147.).

Remarks on the foregoing Example.

30. The example which has occupied us in the last six numbers is not altogether ideal, but is realised to some extent by the motion of a projectile in a void. For if we consider the earth as a sphere, of radius R , and suppose the accelerating force of gravity to vary inversely as the square of the distance r from its centre, and to be $= g$ at the surface, this force will be represented generally by $\frac{g R^2}{r^2}$; and to adapt the differential equations (78.) to the motion of a projectile in a void, it will be sufficient to make

$$U = g R^2 \left(\frac{1}{r} - \frac{1}{R} \right) \dots \dots \dots \dots \dots \quad (173.)$$

If we place the origin of rectangular coordinates at the earth's surface, and suppose the semiaxis of $+z$ to be directed vertically upwards, we shall have

$$r = \sqrt{(R+z)^2 + x^2 + y^2}, \dots \dots \dots \dots \dots \quad (174.)$$

and

$$U = -g z + \frac{g z^2}{R} - \frac{g(x^2 + y^2)}{2R}, \dots \dots \dots \dots \dots \quad (175.)$$

neglecting only those very small terms which have the square of the earth's radius for a divisor: neglecting therefore such terms, the force-function U in this question is of that form (110.) on which all the reasonings of the example have been founded; the small constants μ , ν , being the real and imaginary quantities $\sqrt{\frac{g}{R}}$, $\sqrt{\frac{-2g}{R}}$, respectively. We may therefore apply the results of the recent numbers to the motions of projectiles in a void, by substituting these values for the constants, and altering, where necessary, trigonometrical to exponential functions. But besides the theoretical facility and the little practical importance of researches respecting such projectiles, the results would only be accurate as far as the first negative power (inclusive) of the earth's radius, because the expression (110.) for the force-function U is only accurate so far; and therefore the rigorous and approximate investigations of the six preceding numbers, founded on that expression, are offered only as mathematical illustrations of a general *method*, extending to all problems of dynamics, at least to all those to which the law of living forces applies.

Attracting Systems resumed: Differential Equations of internal or Relative Motion; Integration by the Principal Function.

31. Returning now from this digression on the motion of a single point, to the more important study of an attracting or repelling system, let us resume the differential equations (A.), which may be thus summed up:

$$dt \delta H = \Sigma (d\eta \delta \omega - d\omega \delta \eta); \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (A^2.)$$

and in order to separate the absolute motion of the whole system in space from the motions of its points among themselves, let us choose the following marks of position :

$$x_u = \frac{\Sigma_m x}{\Sigma m}, \quad y_u = \frac{\Sigma_m y}{\Sigma m}, \quad z_u = \frac{\Sigma_m z}{\Sigma m}, \quad \dots \quad \dots \quad \dots \quad (176.)$$

and

$$\xi_i = x_i - x_u, \quad \eta_i = y_i - y_u, \quad \zeta_i = z_i - z_u; \quad \dots \quad \dots \quad \dots \quad (177.)$$

that is, the 3 rectangular coordinates of the centre of gravity of the system, referred to an origin fixed in space, and the $3n-3$ rectangular coordinates of the $n-1$ masses m_1, m_2, \dots, m_{n-1} , referred to the n th mass m_n , as an internal and moveable origin, but to axes parallel to the former. We then find, as in the former Essay,

$$T = \frac{1}{2} (x'_u{}^2 + y'_u{}^2 + z'_u{}^2) \Sigma m + \frac{1}{2} \Sigma_i \cdot m (\xi_i{}^2 + \eta_i{}^2 + \zeta_i{}^2) - \frac{1}{2} \frac{1}{\Sigma m} \{ (\Sigma_i \cdot m \xi_i)^2 + (\Sigma_i \cdot m \eta_i)^2 + (\Sigma_i \cdot m \zeta_i)^2 \}, \quad \dots \quad (178.)$$

the sign of summation Σ_i referring to the first $n-1$ masses only; and therefore,

$$T = \frac{1}{2} \frac{1}{\Sigma m} \left\{ \left(\frac{\delta T}{\delta x'_u} \right)^2 + \left(\frac{\delta T}{\delta y'_u} \right)^2 + \left(\frac{\delta T}{\delta z'_u} \right)^2 \right\} + \frac{1}{2} \Sigma_i \cdot \frac{1}{m} \left\{ \left(\frac{\delta T}{\delta \xi_i} \right)^2 + \left(\frac{\delta T}{\delta \eta_i} \right)^2 + \left(\frac{\delta T}{\delta \zeta_i} \right)^2 \right\} + \frac{1}{2} \frac{1}{m_n} \left\{ \left(\Sigma_i \frac{\delta T}{\delta \xi_i} \right)^2 + \left(\Sigma_i \frac{\delta T}{\delta \eta_i} \right)^2 + \left(\Sigma_i \frac{\delta T}{\delta \zeta_i} \right)^2 \right\}. \quad \dots \quad (179.)$$

If then we put for abridgement,

$$\left. \begin{aligned} x'_i &= \frac{1}{m} \frac{\delta T}{\delta \xi_i} = \xi_i - \frac{\Sigma_i \cdot m \xi_i}{\Sigma m}, \\ y'_i &= \frac{1}{m} \frac{\delta T}{\delta \eta_i} = \eta_i - \frac{\Sigma_i \cdot m \eta_i}{\Sigma m}, \\ z'_i &= \frac{1}{m} \frac{\delta T}{\delta \zeta_i} = \zeta_i - \frac{\Sigma_i \cdot m \zeta_i}{\Sigma m}, \end{aligned} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (180.)$$

we shall have the expression

$$H = \frac{1}{2} (x'_u{}^2 + y'_u{}^2 + z'_u{}^2) \Sigma m + \frac{1}{2} \Sigma_i \cdot m (x'_i{}^2 + y'_i{}^2 + z'_i{}^2) + \frac{1}{2} \frac{1}{m_n} \{ (\Sigma_i \cdot m x'_i)^2 + (\Sigma_i \cdot m y'_i)^2 + (\Sigma_i \cdot m z'_i)^2 \} - U, \quad \dots \quad (B^2.)$$

of which the variation is to be compared with the following form of (A²),

$$dt \delta H = (dx_u \delta x'_u - dx'_u \delta x_u + dy_u \delta y'_u - dy'_u \delta y_u + dz_u \delta z'_u - dz'_u \delta z_u) \Sigma m + \Sigma_i \cdot m (d\xi_i \delta x'_i - dx'_i \delta \xi_i + d\eta_i \delta y'_i - dy'_i \delta \eta_i + d\zeta_i \delta z'_i - dz'_i \delta \zeta_i), \quad \dots \quad (C^2.)$$

in order to form, by our general process, $6n$ differential equations of motion of the first order, between the $6n$ quantities $x_u, y_u, z_u, x'_u, y'_u, z'_u, \xi, \eta, \zeta, x'_i, y'_i, z'_i$ and the time t . In thus taking the variation of H , we are to remember that the force-function U depends only on the $3n-3$ internal coordinates ξ, η, ζ , being of the form

$$U = m_n (m_1 f_1 + m_2 f_2 + \dots + m_{n-1} f_{n-1}) + m_1 m_2 f_{1,2} + m_1 m_3 f_{1,3} + \dots + m_{n-2} m_{n-1} f_{n-2, n-1}, \quad \} \quad (D^2.)$$

in which f_i is a function of the distance of m_i from m_n , and $f_{i,k}$ is a function of the distance of m_i from m_k , such that their derived functions or first differential coefficients, taken with respect to the distances, express the laws of mutual repulsion, being negative in the case of attraction; and then we obtain, as we desired, two separate groups of equations, for the motion of the whole system of points in space, and for the motions of those points among themselves; namely, first, the group

$$\left. \begin{aligned} d x_u &= x'_u dt, \quad d x'_u = 0, \\ d y_u &= y'_u dt, \quad d y'_u = 0, \\ d z_u &= z'_u dt, \quad d z'_u = 0, \end{aligned} \right\} \quad \dots \dots \dots \dots \dots \quad (181.)$$

and secondly the group

$$\left. \begin{aligned} d \xi &= \left(x'_i + \frac{1}{m_n} \sum_i m x'_i \right) dt, \quad d x'_i = \frac{1}{m} \frac{\partial U}{\partial \xi} dt, \\ d \eta &= \left(y'_i + \frac{1}{m_n} \sum_i m y'_i \right) dt, \quad d y'_i = \frac{1}{m} \frac{\partial U}{\partial \eta} dt, \\ d \zeta &= \left(z'_i + \frac{1}{m_n} \sum_i m z'_i \right) dt, \quad d z'_i = \frac{1}{m} \frac{\partial U}{\partial \zeta} dt. \end{aligned} \right\} \quad \dots \dots \dots \quad (182.)$$

The six differential equations of the first order, (181.), between $x_u, y_u, z_u, x'_u, y'_u, z'_u$ and t , contain the law of rectilinear and uniform motion of the centre of gravity of the system; and the $6n - 6$ equations of the same order, (182.), between the $6n - 6$ variables $\xi, \eta, \zeta, x', y', z'$, and the time, are forms for the differential equations of internal or relative motion. We might eliminate the $3n - 3$ auxiliary variables x', y', z' , between these last equations, and so obtain the following other group of $3n - 3$ equations of the second order, involving only the relative coordinates and the time,

$$\left. \begin{aligned} \xi'' &= \frac{1}{m} \frac{\partial U}{\partial \xi} + \frac{1}{m_n} \sum_i \frac{\partial U}{\partial \xi}, \\ \eta'' &= \frac{1}{m} \frac{\partial U}{\partial \eta} + \frac{1}{m_n} \sum_i \frac{\partial U}{\partial \eta}, \\ \zeta'' &= \frac{1}{m} \frac{\partial U}{\partial \zeta} + \frac{1}{m_n} \sum_i \frac{\partial U}{\partial \zeta}; \end{aligned} \right\} \quad \dots \dots \dots \quad (183.)$$

but it is better for many purposes to retain them under the forms (182.), omitting, however, for simplicity, the lower accents of the auxiliary variables x', y', z' , because it is easy to prove that these auxiliary variables (180.) are the components of centrobaric velocity, and because, in investigating the properties of internal or relative motion, we are at liberty to suppose that the centre of gravity of the system is fixed in space, at the origin of $x y z$. We may also, for simplicity, omit the lower accent of Σ , understanding that the summations are to fall only on the first $n - 1$ masses, and denoting for greater distinctness the n th mass by a separate symbol M ; and then we

may comprise the differential equations of relative motion in the following simplified formula,

$$dt \delta H = \Sigma . m (d\xi \delta x' - dx' \delta \xi + d\eta \delta y' - dy' \delta \eta + d\zeta \delta z' - dz' \delta \zeta), \quad \dots \quad (E^2.)$$

in which

$$H = \frac{1}{2} \Sigma . m (x'^2 + y'^2 + z'^2) + \frac{1}{2M} \{ (\Sigma . m x')^2 + (\Sigma . m y')^2 + (\Sigma . m z')^2 \} - U. \quad (F^2.)$$

And the integrals of these equations of relative motion are contained (by our general method) in the formula

$$dS = \Sigma . m (x' \delta \xi - a' \delta \alpha + y' \delta \eta - b' \delta \beta + z' \delta \zeta - c' \delta \gamma), \quad \dots \quad (G^2.)$$

in which $\alpha \beta \gamma a' b' c'$ denote the initial values of $\xi \eta \zeta x' y' z'$, and S is the *principal function of relative motion* of the system; that is, the former function S , simplified by the omission of the part which vanishes when the centre of gravity is fixed, and which gives in general the laws of motion of that centre, or the integrals of the equations (181.).

Second Example: Case of a Ternary or Multiple System with one Predominant Mass;

Equations of the undisturbed motions of the other masses about this, in their several Binary Systems; Differentials of all their Elements, expressed by the coefficients of one Disturbing Function.

32. Let us now suppose that the $n - 1$ masses m are small in comparison with the n th mass M ; and let us separate the expression $(F^2.)$ for H into the two following parts,

$$\left. \begin{aligned} H_1 &= \Sigma . \frac{m}{2} \left(1 + \frac{m}{M} \right) (x'^2 + y'^2 + z'^2) - M \Sigma . m f, \\ H_2 &= \frac{m_1 m_2}{M} (x'_1 x'_2 + y'_1 y'_2 + z'_1 z'_2 - M f_{1,2}) + \dots \\ &\quad + \frac{m_i m_k}{M} (x'_i x'_k + y'_i y'_k + z'_i z'_k - M f_{i,k}) + \dots, \end{aligned} \right\} \quad \dots \quad (H^2.)$$

of which the latter is small in comparison with the former, and may be neglected in a first approximation. Suppressing it accordingly, we are conducted to the following $6n - 6$ differential equations of the 1st order, belonging to a simpler motion, which may be called the *undisturbed*:

$$\left. \begin{aligned} \frac{d\xi}{dt} &= \frac{1}{m} \frac{\delta H_1}{\delta x'} = \left(1 + \frac{m}{M} \right) x', \quad \frac{dx'}{dt} = - \frac{1}{m} \frac{\delta H_1}{\delta \xi} = M \frac{\delta f}{\delta \xi}; \\ \frac{d\eta}{dt} &= \frac{1}{m} \frac{\delta H_1}{\delta y'} = \left(1 + \frac{m}{M} \right) y', \quad \frac{dy'}{dt} = - \frac{1}{m} \frac{\delta H_1}{\delta \eta} = M \frac{\delta f}{\delta \eta}; \\ \frac{d\zeta}{dt} &= \frac{1}{m} \frac{\delta H_1}{\delta z'} = \left(1 + \frac{m}{M} \right) z', \quad \frac{dz'}{dt} = - \frac{1}{m} \frac{\delta H_1}{\delta \zeta} = M \frac{\delta f}{\delta \zeta}. \end{aligned} \right\} \quad \dots \quad (I^2.)$$

These equations arrange themselves in $n - 1$ groups, corresponding to the $n - 1$ binary systems (m, M) ; and it is easy to integrate the equations of each group separately. We may suppose, then, these integrals found, under the forms,

$$\left. \begin{aligned} x &= \chi^{(1)}(t, \xi, \eta, \zeta, x', y', z'), & v &= \chi^{(4)}(t, \xi, \eta, \zeta, x', y', z'), \\ \lambda &= \chi^{(2)}(t, \xi, \eta, \zeta, x', y', z'), & \tau &= \chi^{(5)}(t, \xi, \eta, \zeta, x', y', z'), \\ \mu &= \chi^{(3)}(t, \xi, \eta, \zeta, x', y', z'), & \omega &= \chi^{(6)}(t, \xi, \eta, \zeta, x', y', z'), \end{aligned} \right\} \quad \dots \quad (K^2.)$$

the six quantities $\alpha, \lambda, \mu, \nu, \tau, \omega$ being constant for the undisturbed motion of any one binary system; and therefore the six functions $\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \chi^{(4)}, \chi^{(5)}, \chi^{(6)}$, or $x, \lambda, \mu, \nu, \tau, \omega$, being such as to satisfy *identically* the following equation,

$$0 = m \frac{\delta x}{\delta t} + \frac{\delta x \delta H_1}{\delta \xi \delta x'} - \frac{\delta x \delta H_1}{\delta x' \delta \xi} + \frac{\delta x \delta H_1}{\delta \eta \delta y'} - \frac{\delta x \delta H_1}{\delta y' \delta \eta} + \frac{\delta x \delta H_1}{\delta \zeta \delta z'} - \frac{\delta x \delta H_1}{\delta z' \delta \zeta}, \quad (L^2.)$$

with five other equations analogous, for the five other elements $\lambda, \mu, \nu, \tau, \omega$, in any one binary system (m, M).

33. Returning now to the original multiple system, we may retain as definitions the equations ($K^2.$), but then we can no longer consider the elements $x_i \lambda_i \mu_i \nu_i \tau_i \omega_i$ of the binary system (m_i, M_i) as constant, because this system is now disturbed by the other masses m_k ; however, the $6n - 6$ equations of disturbed relative motion, when put under the forms

$$\left. \begin{aligned} m \frac{d\xi}{dt} &= \frac{\delta H_1}{\delta x'} + \frac{\delta H_2}{\delta x'}, & m \frac{dx'}{dt} &= -\frac{\delta H_1}{\delta \xi} - \frac{\delta H_2}{\delta \xi}, \\ m \frac{d\eta}{dt} &= \frac{\delta H_1}{\delta y'} + \frac{\delta H_2}{\delta y'}, & m \frac{dy'}{dt} &= -\frac{\delta H_1}{\delta \eta} - \frac{\delta H_2}{\delta \eta}, \\ m \frac{d\zeta}{dt} &= \frac{\delta H_1}{\delta z'} + \frac{\delta H_2}{\delta z'}, & m \frac{dz'}{dt} &= -\frac{\delta H_1}{\delta \zeta} - \frac{\delta H_2}{\delta \zeta}, \end{aligned} \right\} \dots \quad (M^2.)$$

and combined with the identical equations of the kind ($L^2.$), give the following simple expression for the differential of the element x , in its disturbed and variable state,

$$m \frac{dx}{dt} = \frac{\delta x \delta H_2}{\delta \xi \delta x'} - \frac{\delta x \delta H_2}{\delta x' \delta \xi} + \frac{\delta x \delta H_2}{\delta \eta \delta y'} - \frac{\delta x \delta H_2}{\delta y' \delta \eta} + \frac{\delta x \delta H_2}{\delta \zeta \delta z'} - \frac{\delta x \delta H_2}{\delta z' \delta \zeta}, \quad (N^2.)$$

together with analogous expressions for the differentials of the other elements. And if we express $\xi \eta \zeta x' y' z'$, and therefore H_2 itself, as depending on the time and on these varying elements, we may transform the $6n - 6$ differential equations of the 1st order, ($M^2.$), between $\xi \eta \zeta x' y' z' t$, into the same number of equations of the same order between the varying elements and the time; which will be of the forms

$$\left. \begin{aligned} m \frac{dx}{dt} &= \{z, \lambda\} \frac{\delta H_2}{\delta \lambda} + \{z, \mu\} \frac{\delta H_2}{\delta \mu} + \{z, \nu\} \frac{\delta H_2}{\delta \nu} + \{z, \tau\} \frac{\delta H_2}{\delta \tau} + \{z, \omega\} \frac{\delta H_2}{\delta \omega}, \\ m \frac{d\lambda}{dt} &= \{\lambda, z\} \frac{\delta H_2}{\delta x} + \{\lambda, \mu\} \frac{\delta H_2}{\delta \mu} + \{\lambda, \nu\} \frac{\delta H_2}{\delta \nu} + \{\lambda, \tau\} \frac{\delta H_2}{\delta \tau} + \{\lambda, \omega\} \frac{\delta H_2}{\delta \omega}, \\ m \frac{d\mu}{dt} &= \{\mu, z\} \frac{\delta H_2}{\delta x} + \{\mu, \lambda\} \frac{\delta H_2}{\delta \lambda} + \{\mu, \nu\} \frac{\delta H_2}{\delta \nu} + \{\mu, \tau\} \frac{\delta H_2}{\delta \tau} + \{\mu, \omega\} \frac{\delta H_2}{\delta \omega}, \\ m \frac{d\nu}{dt} &= \{\nu, z\} \frac{\delta H_2}{\delta x} + \{\nu, \lambda\} \frac{\delta H_2}{\delta \lambda} + \{\nu, \mu\} \frac{\delta H_2}{\delta \mu} + \{\nu, \tau\} \frac{\delta H_2}{\delta \tau} + \{\nu, \omega\} \frac{\delta H_2}{\delta \omega}, \\ m \frac{d\tau}{dt} &= \{\tau, z\} \frac{\delta H_2}{\delta x} + \{\tau, \lambda\} \frac{\delta H_2}{\delta \lambda} + \{\tau, \mu\} \frac{\delta H_2}{\delta \mu} + \{\tau, \nu\} \frac{\delta H_2}{\delta \nu} + \{\tau, \omega\} \frac{\delta H_2}{\delta \omega}, \\ m \frac{d\omega}{dt} &= \{\omega, z\} \frac{\delta H_2}{\delta x} + \{\omega, \lambda\} \frac{\delta H_2}{\delta \lambda} + \{\omega, \mu\} \frac{\delta H_2}{\delta \mu} + \{\omega, \nu\} \frac{\delta H_2}{\delta \nu} + \{\omega, \tau\} \frac{\delta H_2}{\delta \tau}, \end{aligned} \right\} \quad (O^2.)$$

if we put, for abridgement,

$$\{z, \lambda\} = \frac{\delta x \delta \lambda}{\delta \xi \delta x'} - \frac{\delta x \delta \lambda}{\delta x' \delta \xi} + \frac{\delta x \delta \lambda}{\delta \eta \delta y'} - \frac{\delta x \delta \lambda}{\delta y' \delta \eta} + \frac{\delta x \delta \lambda}{\delta \zeta \delta z'} - \frac{\delta x \delta \lambda}{\delta z' \delta \zeta} \quad \dots \quad (P^2)$$

and form the other symbols $\{z, \mu\}$, $\{\lambda, z\}$, &c., from this, by interchanging the letters. It is evident that these symbols have the properties,

$$\{\lambda, z\} = -\{z, \lambda\}, \{z, z\} = 0; \dots \dots \dots \dots \dots \quad (184.)$$

and it results from the principles of the 15th number, that these combinations $\{z, \lambda\}$, &c., when expressed as functions of the elements, do not contain the time explicitly. There are in general, by (184.), only 15 such distinct combinations for each of the $n - 1$ binary systems; but there would thus be, in all, $15n - 15$, if they admitted of no further reductions: however, it results from the principles of the 16th number, that $12n - 12$ of these combinations may be made to vanish by a suitable choice of the elements. The following is another way of effecting as great a simplification, at least for that extensive class of cases in which the undisturbed distance between the two points of each binary system (m, M) admits of a minimum value.

Simplification of the Differential Expressions by a suitable choice of the Elements.

34. When the undisturbed distance r of m from M admits of such a minimum q , corresponding to a time τ , and satisfying at that time the conditions

$$r' = 0, r'' > 0, \dots \dots \dots \dots \dots \dots \dots \dots \quad (185.)$$

then the integrals of the group (I^2) , or the known rules of the undisturbed motion of m about M , may be presented in the following manner :

$$z = \sqrt{(\xi y' - \eta x')^2 + (\eta z' - \zeta y')^2 + (\zeta x' - \xi z')^2};$$

$$\lambda = z - \xi y' + \eta x';$$

$$\mu = \frac{M+m}{2M} (x'^2 + y'^2 + z'^2) - Mf(r);$$

$$\nu = \tan^{-1} \cdot \frac{\eta z' - \zeta y'}{\xi z' - \xi x'};$$

$$\tau = t - \int_q^r \frac{\sqrt{\frac{M}{M+m}} \cdot \frac{dr}{\sqrt{dr^3}} \cdot dr}{\sqrt{\left\{ \frac{2\mu}{M+m} + 2Mf(r) - \left(1 + \frac{m}{M}\right) \frac{x^3}{r^3} \right\}}};$$

$$\omega = \nu + \sin^{-1} \cdot \frac{x \xi r^{-1}}{\sqrt{2\lambda x - \lambda^3}} - \int_q^r \frac{\sqrt{\frac{M+m}{M}} \cdot \frac{dr}{\sqrt{dr^3}} \cdot \frac{x}{r^3} \cdot dr}{\sqrt{\left\{ \frac{2\mu}{M+m} + 2Mf(r) - \left(1 + \frac{m}{M}\right) \frac{x^3}{r^3} \right\}}};$$

the minimum distance q being a function of the two elements z, μ , which must satisfy the conditions

$$2\mu + 2Mf(q) - \left(1 + \frac{m}{M}\right) \frac{x^3}{q^3} = 0, Mf'(q) + \left(1 + \frac{m}{M}\right) \frac{x^2}{q^3} > 0; \quad (186.)$$

and $\sin^{-1} s, \tan^{-1} t$, being used (according to Sir JOHN HERSCHEL's notation) to ex-

press, *not* the cosecant and cotangent, but the *inverse functions* corresponding to sine and cosine, or the arcs which are more commonly called arc ($\sin = s$), arc ($\tan = t$).

It must also be observed that the factor $\frac{dr}{\sqrt{dr^2}}$, which we have introduced under the signs of integration, is not superfluous, but is designed to be taken as equal to positive or negative unity, according as dr is positive or negative; that is, according as r is increasing or diminishing, so as to make the element under each integral sign constantly positive. In general, it appears to be a useful rule, though not always followed by analysts, to employ the real radical symbol \sqrt{R} only for positive quantities, unless the negative sign be expressly prefixed; and then $\frac{r}{\sqrt{r^2}}$ will denote positive or negative unity, according as r is positive or negative. The arc given by its sine, in the expression of the element ω , is supposed to be so chosen as to increase continually with the time.

35. After these remarks on the notation, let us apply the formula (P².) to calculate the values of the 15 combinations such as $\{z, \lambda\}$, of the 6 constants or elements (Q².).

Since

$$r = \sqrt{(\xi^2 + \eta^2 + \zeta^2)}, \dots \dots \dots \dots \dots \dots \quad (187.)$$

it is easy to perceive that the six combinations of the 4 first elements are as follows:

$$\{z, \lambda\} = 0, \{z, \mu\} = 0, \{z, \nu\} = 0, \{\lambda, \mu\} = 0, \{\lambda, \nu\} = 1, \{\mu, \nu\} = 0. \dots \quad (188.)$$

To form the 4 combinations of these 4 first elements with τ , we may observe, that this 5th element τ , as expressed in (Q².), involves explicitly (besides the time) the distance r , and the two elements z, μ ; but the combinations already determined show that these two elements may be treated as constant in forming the four combinations now sought; we need only attend, therefore, to the variation of r , and if we interpret by the rule (P².¹) the symbols $\{z, r\} \{\lambda, r\} \{\mu, r\} \{\nu, r\}$, and attend to the equations (I².), we see that

$$\{z, r\} = 0, \{\lambda, r\} = 0, \{\mu, r\} = -\frac{dr}{dt}, \{\nu, r\} = 0, \dots \dots \dots \quad (189.)$$

$\frac{dr}{dt}$ being the total differential coefficient of r in the undisturbed motion, as determined by the equations (I².); and, therefore, that

$$\{z, \tau\} = 0, \{\lambda, \tau\} = 0, \{\nu, \tau\} = 0, \dots \dots \dots \quad (190.)$$

and

$$\{\mu, \tau\} = -\frac{\delta \tau}{\delta r} \frac{dr}{dt} = +\frac{dt}{dr} \frac{dr}{dt} = 1: \dots \dots \dots \quad (191.)$$

observing that in differentiating the expressions of the elements (Q².), we may treat those elements as constant, if we change the differentials of $\xi \eta \zeta x' y' z'$ to their undisturbed values. It remains to calculate the 5 combinations of these 5 elements with the last element ω ; which is given by (Q².²) as a function of the distance r , the coordinate ζ , and the 4 elements z, λ, μ, ν ; so that we may employ this formula,

$$\{e, \omega\} = \frac{\delta \omega}{\delta r} \{e, r\} + \frac{\delta \omega}{\delta \zeta} \{e, \zeta\} + \frac{\delta \omega}{\delta x} \{e, z\} + \frac{\delta \omega}{\delta \lambda} \{e, \lambda\} + \frac{\delta \omega}{\delta \mu} \{e, \mu\} + \frac{\delta \omega}{\delta \nu} \{e, \nu\}, \quad (192.)$$

in which, if e be any of the first five elements, or the distance r ,

$$\{e, r\} = -\frac{1}{r} \left(\xi \frac{\delta e}{\delta x'} + \eta \frac{\delta e}{\delta y'} + \zeta \frac{\delta e}{\delta z'} \right), \{e, \zeta\} = -\frac{\delta e}{\delta z'}, \{e, z\} = 0, \quad (193.)$$

and

$$\frac{\delta \omega}{\delta \xi} = \left(\frac{\delta x}{\delta z'} \right)^{-1}, \quad \frac{\delta \omega}{\delta r} = -\frac{d\xi}{dr} \frac{\delta \omega}{\delta \xi}, \quad \frac{\delta \omega}{\delta \eta} = 1; \quad \dots \dots \dots \quad (194.)$$

the formula (192.) may therefore be thus written:

$$\left. \begin{aligned} \{e, \omega\} &= \left\{ \frac{z' \left(\xi \frac{\delta e}{\delta x'} + \eta \frac{\delta e}{\delta y'} + \zeta \frac{\delta e}{\delta z'} \right) - \frac{\delta e}{\delta z'}}{\xi x' + \eta y' + \zeta z'} \right\} \left(\frac{\delta x}{\delta z'} \right)^{-1} \\ &\quad + \{e, r\} + \frac{\delta \omega}{\delta \lambda} \{e, \lambda\} + \frac{\delta \omega}{\delta \mu} \{e, \mu\}. \end{aligned} \right\} \quad \dots \dots \dots \quad (195.)$$

We easily find, by this formula, that

$$\{z, \omega\} = -1; \quad \{\lambda, \omega\} = 0; \quad \{\mu, \omega\} = 0; \quad \{r, \omega\} = \frac{dr}{dt} \frac{\delta \omega}{\delta \mu}; \quad \dots \dots \quad (196.)$$

and

$$\{v, \omega\} = -\frac{\delta \eta}{\delta z'} \frac{\delta \omega}{\delta \xi} - \frac{\delta \omega}{\delta \lambda} = 0. \quad \dots \dots \dots \quad (197.)$$

The formula (195.) extends to the combination $\{\tau, \omega\}$ also; but in calculating this last combination we are to remember that τ is given by (Q²) as a function of z, μ, r , such that

$$\frac{\delta \tau}{\delta r} = -\frac{dt}{dr}; \quad \dots \dots \dots \quad (198.)$$

and thus we see, with the help of the combinations (196.) already determined, that

$$\{\tau, \omega\} = -\frac{\delta \tau}{\delta x} - \frac{\delta \omega}{\delta \mu} = \frac{\delta}{\delta x} \int_q^r \Theta_r dr + \frac{\delta}{\delta \mu} \int_q^r \Omega_r dr, \quad \dots \dots \dots \quad (199.)$$

if we represent for abridgement by Θ_r and Ω_r the coefficients of dr under the integral signs in (Q²), namely,

$$\left. \begin{aligned} \Theta_r &= \sqrt{\frac{M}{M+m} \frac{dr}{\sqrt{dr^2}}} \left\{ 2\mu + 2Mf(r) - \frac{M+m}{M} \cdot \frac{x^2}{r^2} \right\}^{-\frac{1}{2}} \\ \Omega_r &= \frac{x}{r^2} \sqrt{\frac{M+m}{M} \frac{dr}{\sqrt{dr^2}}} \left\{ 2\mu + 2Mf(r) - \frac{M+m}{M} \cdot \frac{x^2}{r^2} \right\}^{-\frac{1}{2}} \end{aligned} \right\} \quad \dots \quad (200.)$$

These coefficients are evidently connected by the relation

$$\frac{\delta \Theta_r}{\delta x} + \frac{\delta \Omega_r}{\delta \mu} = 0, \quad \dots \dots \dots \quad (201.)$$

which gives

$$\frac{\delta}{\delta x} \int_{r_i}^r \Theta_r dr + \frac{\delta}{\delta \mu} \int_{r_i}^r \Omega_r dr = 0, \quad \dots \dots \dots \quad (202.)$$

r_i being any quantity which does not vary with the elements x and μ ; we might therefore at once conclude by (199.) that the combination $\{\tau, \omega\}$ vanishes, if a diffi-

culty were not occasioned by the necessity of varying the lower limit q , which depends on those two elements, and by the circumstance that at this lower limit the coefficients Θ_r, Ω_r become infinite. However, the relation (202.) shows that we may express this combination $\{\tau, \omega\}$ as follows:

$$\{\tau, \omega\} = \frac{\delta}{\delta z} \int_q^{r'} \Theta_r dr + \frac{\delta}{\delta \mu} \int_q^{r'} \Omega_r dr, \quad \dots \quad (203.)$$

r , being an auxiliary and arbitrary quantity, which cannot really affect the result, but may be made to facilitate the calculation; or in other words, we may assign to the distance r any arbitrary value, not varying for infinitesimal variations of z, μ , which may assist in calculating the value of the expression (199.). We may therefore suppose that the increase of distance $r - q$ is small, and corresponds to a small positive interval of time $t - \tau$, during which the distance r and its differential coefficient r' are constantly increasing; and then after the first moment τ , the quantity

$$\Theta_r = \frac{1}{r'} \quad \dots \quad (204.)$$

will be constantly finite, positive, and decreasing, during the same interval, so that its integral must be greater than if it had constantly its final value; that is,

$$t - \tau = \int_q^{r'} \Theta_r dr > (r - q) \Theta_r. \quad \dots \quad (205.)$$

Hence, although Θ_r tends to infinity, yet $(r - q) \Theta_r$ tends to zero, when by diminishing the interval we make r tend to q ; and therefore the following difference

$$\int_q^{r'} \Omega_r dr - \frac{M+m}{M} \frac{z}{q^2} \int_q^{r'} \Theta_r dr = \frac{M+m}{M} \int_q^{r'} \left(\frac{z}{r^2} - \frac{z}{q^2} \right) \Theta_r dr, \quad \dots \quad (206.)$$

will also tend to 0, and so will also its partial differential coefficient of the first order, taken with respect to μ . We find therefore the following formula for $\{\tau, \omega\}$, (remembering that this combination has been shown to be independent of r),

$$\{\tau, \omega\} = \frac{\Delta}{r - q} \left\{ \frac{\delta}{\delta z} \int_q^{r'} \Theta_r dr + \frac{M+m}{M} \frac{z}{q^2} \frac{\delta}{\delta \mu} \int_q^{r'} \Theta_r dr \right\}; \quad \dots \quad (207.)$$

the sign $\frac{\Delta}{r - q}$ implying that the limit is to be taken to which the expression tends when r tends to q . In this last formula, as in (199.), the integral $\int_q^{r'} \Theta_r dr$ may be considered as a known function of r, q, z, μ , or simply of r, q, z , if μ be eliminated by the first condition (186.); and since it vanishes independently of z when $r = q$, it may be thus denoted:

$$\int_q^{r'} \Theta_r dr = \varphi(r, q, z) - \varphi(q, q, z), \quad \dots \quad (208.)$$

the form of the function φ depending on the law of attraction or repulsion. This integral therefore, when considered as depending on z and μ , by depending on z and q , need not be varied with respect to z , in calculating $\{\tau, \omega\}$ by (207.), because

its partial differential coefficient $\left(\frac{\delta}{\delta x_n} \int_q^r \Theta_r dr\right)$, obtained by treating q as constant, vanishes at the limit $r = q$; nor need it be varied with respect to q , because, by (186.),

$$\frac{\partial q}{\partial x} + \frac{M+m}{M} \frac{x}{q^3} \frac{\partial q}{\partial \mu} = 0; \quad \dots \dots \dots \dots \dots \dots \dots \quad (209.)$$

it may therefore be treated as constant, and we find at last

$$\{\tau, \omega\} = 0, \quad \dots \quad (210.)$$

the two terms (199.) or (203.) both tending to infinity when r tends to q , but always destroying each other.

36. Collecting now our results, and presenting for greater clearness each combination under the two forms in which it occurs when the order of the elements is changed, we have, for each binary system, the following thirty expressions :

$$\left. \begin{array}{l} \{z, \lambda\} = 0, \{z, \mu\} = 0, \{z, v\} = 0, \{z, \tau\} = 0, \{z, \omega\} = -1, \\ \{\lambda, z\} = 0, \{\lambda, \mu\} = 0, \{\lambda, v\} = 1, \{\lambda, \tau\} = 0, \{\lambda, \omega\} = 0, \\ \{\mu, z\} = 0, \{\mu, \lambda\} = 0, \{\mu, v\} = 0, \{\mu, \tau\} = 1, \{\mu, \omega\} = 0, \\ \{v, z\} = 0, \{v, \lambda\} = -1, \{v, \mu\} = 0, \{v, \tau\} = 0, \{v, \omega\} = 0, \\ \{\tau, z\} = 0, \{\tau, \lambda\} = 0, \{\tau, \mu\} = -1, \{\tau, v\} = 0, \{\tau, \omega\} = 0, \\ \{\omega, z\} = 1, \{\omega, \lambda\} = 0, \{\omega, \mu\} = 0, \{\omega, v\} = 0, \{\omega, \tau\} = 0; \end{array} \right\} \quad \cdot \quad (R^2).$$

so that the three combinations

$$\{\mu, \tau\} \quad \{\omega, \pi\} \quad \{\lambda, \nu\}$$

are each equal to positive unity; the three inverse combinations

$$\{\tau, \mu\} \quad \{\pi, \omega\} \quad \{v, \lambda\}$$

are each equal to negative unity; and all the others vanish. The six differential equations of the first order, for the 6 varying elements of any one binary system (m , M), are therefore, by (O²),

$$\left. \begin{aligned} m \frac{d\mu}{dt} &= \frac{\delta H_3}{\delta \tau}, \quad m \frac{d\tau}{dt} = -\frac{\delta H_2}{\delta \mu}, \\ m \frac{d\omega}{dt} &= \frac{\delta H_2}{\delta x}, \quad m \frac{dx}{dt} = -\frac{\delta H_1}{\delta \omega}, \\ m \frac{d\lambda}{dt} &= \frac{\delta H_1}{\delta y}, \quad m \frac{dy}{dt} = -\frac{\delta H_2}{\delta \lambda}; \end{aligned} \right\} \dots \quad (S^2).$$

and, if we still omit the variation of t , they may all be summed up in this form for the variation of H_2 ,

$$\delta H_2 = \Sigma \cdot m (\mu' \delta \tau - \tau' \delta \mu + \omega' \delta \kappa - \kappa' \delta \omega + \lambda' \delta \nu - \nu' \delta \lambda), \quad . \quad (T^2)$$

which single formula enables us to derive all the $6n - 6$ differential equations of the first order, for all the varying elements of all the binary systems, from the variation or from the partial differential coefficients of a single quantity H_2 , expressed as a function of those elements.

If we choose to introduce into the expression (T².), for δH_2 , the variation of the time t , we have only to change $\delta \tau$ to $\delta \tau - \delta t$, because, by (Q²), δt enters only so accompanied; that is, t enters only under the form $t - \tau_i$ in the expressions of $\xi_i \eta_i \zeta_i x'_i y'_i z'_i$ as functions of the time and of the elements; we have, therefore,

$$\frac{\delta H_2}{\delta t} = - \sum \frac{\delta H_2}{\delta \tau} = - \sum m \mu'; \quad \dots \dots \dots \dots \quad (211.)$$

and since, by (H²), (Q²),

$$H_1 = \sum m \mu, \quad \dots \dots \dots \dots \quad (212.)$$

we find finally,

$$\frac{d H_1}{d t} = - \frac{\delta H_2}{\delta t}. \quad \dots \dots \dots \dots \quad (U^2.)$$

This remarkable form for the differential of H_1 , considered as a varying element, is general for all problems of dynamics. It may be deduced by the general method from the formulæ of the 13th and 14th numbers, which give

$$\left. \begin{aligned} \frac{d H_1}{d t} &= \frac{\delta H_2}{\delta x_1} \sum \left(\frac{\delta H_1}{\delta \eta} \frac{\delta x_1}{\delta \omega} - \frac{\delta H_1}{\delta \omega} \frac{\delta x_1}{\delta \eta} \right) + \dots + \frac{\delta H_2}{\delta x_{6n}} \sum \left(\frac{\delta H_1}{\delta \eta} \frac{\delta x_{6n}}{\delta \omega} - \frac{\delta H_1}{\delta \omega} \frac{\delta x_{6n}}{\delta \eta} \right) \\ &= \frac{\delta H_2}{\delta x_1} \frac{\delta x_1}{\delta t} + \frac{\delta H_2}{\delta x_2} \frac{\delta x_2}{\delta t} + \dots + \frac{\delta H_2}{\delta x_{6n}} \frac{\delta x_{6n}}{\delta t} = - \frac{\delta H_2}{\delta t}, \end{aligned} \right\} \quad (213.)$$

$x_1 x_2 \dots x_{6n}$ being any $6n$ elements of a system expressed as functions of the time and of the quantities $\eta \omega$; or more concisely by this special consideration, that $H_1 + H_2$ is constant in the disturbed motion, and that in taking the first total differential coefficient of H_2 with respect to the time, the elements may by (F¹.) be treated as constant. It is also a remarkable corollary of the general principles just referred to, but one not difficult to verify, that the first partial differential coefficient $\frac{\delta x_s}{\delta t}$ of any element x_s , taken with respect to the time, may be expressed as a function of the elements alone, not involving the time explicitly.

On the essential distinction between the Systems of Varying Elements considered in this Essay and those hitherto employed by mathematicians.

37. When we shall have integrated the differential equations of varying elements (S².), we can then calculate the varying relative coordinates $\xi \eta \zeta$ for any binary system (m, M), by the rules of undisturbed motion, as expressed by the equations (I².), (Q²), or by the following connected formulæ :

$$\left. \begin{aligned} \xi &= r \left(\cos \theta + \frac{\lambda}{x} \sin (\theta - \nu) \sin \nu \right), \\ \eta &= r \left(\sin \theta - \frac{\lambda}{x} \sin (\theta - \nu) \cos \nu \right), \\ \zeta &= \frac{r}{x} \sqrt{2 \lambda z - \lambda^2} \sin (\theta - \nu); \end{aligned} \right\} \quad \dots \dots \dots \dots \quad (V^2.)$$

in which the distance r is determined as a function of the time t and of the elements τ, z, μ , by the 5th equation (Q²), and in which

$$\theta = \omega + \int_q^r \frac{\sqrt{\frac{M+m}{M} \cdot \frac{dr}{\sqrt{dr^2 - \frac{2\mu}{M} \cdot \frac{z^2}{r^3}}}} \cdot \frac{z}{r^2} dr}{\sqrt{\left\{ 2\mu + 2Mf(r) - \frac{M+m}{M} \cdot \frac{z^2}{r^3} \right\}}}, \dots \quad (W^2)$$

q being still the minimum of r , when the orbit is treated as constant, and being still connected with the elements z, μ , by the first equation of condition (186.). In astronomical language, M is the sun, m a planet, $\xi \eta \zeta$ are the heliocentric rectangular coordinates, r is the radius vector, θ the longitude in the orbit, ω the longitude of the perihelion, ν of the node, $\theta - \omega$ is the true anomaly, $\theta - \nu$ the argument of latitude, μ the constant part of the half square of undisturbed heliocentric velocity, diminished in the ratio of the sun's mass (M) to the sum ($M + m$) of masses of sun and planet, z is the double of the areal velocity diminished in the same ratio, $\frac{z}{r}$ is the versed sine of the inclination of the orbit, q the perihelion distance, and τ the time of perihelion passage. The law of attraction or repulsion is here left undetermined; for NEWTON's law, μ is the sun's mass divided by the axis major of the orbit taken negatively, and z is the square root of the semiparameter, multiplied by the sun's mass, and divided by the square root of the sum of the masses of sun and planet. But the varying ellipse or other orbit, which the foregoing formulæ require, differs essentially (though little) from that hitherto employed by astronomers: because it gives correctly the heliocentric coordinates, but *not* the heliocentric components of velocity, without differentiating the elements in the calculation; and therefore does *not touch*, but *cuts*, (though under a very small angle,) the actual heliocentric orbit, described under the influence of all the disturbing forces.

38. For it results from the foregoing theory, that if we differentiate the expressions (V².) for the heliocentric coordinates, without differentiating the elements, and then assign to those new varying elements their values as functions of the time, obtained from the equations (S².), and deduce the centrobaric components of velocity by the formulæ (I².), or by the following:

$$x' = \frac{M\xi}{M+m}, \quad y' = \frac{M\eta}{M+m}, \quad z' = \frac{M\zeta}{M+m}; \quad \dots \quad (214.)$$

then these centrobaric components will be the same functions of the time and of the new varying elements which might be otherwise deduced by elimination from the integrals (Q²), and will represent rigorously (by the extension given in the theory to those last-mentioned integrals) the components of velocity of the disturbed planet m , relatively to the centre of gravity of the whole solar system. We chose, as more suitable to the general course of our method, that these centrobaric components of velocity should be the auxiliary variables to be combined with the heliocentric coordinates, and to have their disturbed values rigorously expressed by the formulæ

of undisturbed motion ; but in making this choice it became necessary to modify these latter formulae, and to determine a varying orbit essentially distinct in theory (though little differing in practice) from that conceived so beautifully by LAGRANGE. The orbit which he imagined was more simply connected with the heliocentric motion of a *single planet*, since it gave, for such heliocentric motion, the velocity as well as the position ; the orbit which we have chosen is perhaps more closely combined with the conception of a *multiple system*, moving about its common centre of gravity, and influenced in every part by the actions of all the rest. Whichever orbit shall be hereafter adopted by astronomers, they will remember that both are equally fit to represent the celestial appearances, if the numeric elements of either set be suitably determined by observation, and the elements of the other set of orbits be deduced from these by calculation. Meantime mathematicians will judge, whether in sacrificing a part of the simplicity of that geometrical conception on which the theories of LAGRANGE and POISSON are founded, a simplicity of another kind has not been introduced, which was wanting in those admirable theories ; by our having succeeded in expressing rigorously the differentials of *all* our own new varying elements through the coefficients of a *single* function : whereas it has seemed necessary hitherto to employ one function for the Earth disturbed by Venus, and another function for Venus disturbed by the Earth.

Integration of the Simplified Equations, which determine the new varying Elements.

39. The simplified differential equations of varying elements, (S²), are of the same form as the equations (A.), and may be integrated in a similar manner. If we put, for abridgement,

$$(\tau, z, v) = \int_0^t \left\{ \sum \left(\tau \frac{\delta H_2}{\delta \tau} + z \frac{\delta H_2}{\delta z} + v \frac{\delta H_2}{\delta v} \right) - H_2 \right\} dt, \quad \dots \quad (X^2.)$$

and interpret similarly the symbols (μ, ω, λ), &c., we can easily assign the variations of the following 8 combinations, (τ, z, v) (μ, ω, λ) (μ, z, v) (τ, ω, λ) (τ, ω, v) (μ, z, λ) (μ, z, v) ; namely,

$$\begin{aligned} \delta(\tau, z, v) &= \Sigma . m (\tau \delta \mu - \tau_0 \delta \mu_0 + z \delta \omega - z_0 \delta \omega_0 + v \delta \lambda - v_0 \delta \lambda_0) - H_2 \delta t, \\ \delta(\mu, \omega, \lambda) &= \Sigma . m (\mu_0 \delta \tau_0 - \mu \delta \tau + \omega_0 \delta z_0 - \omega \delta z + \lambda_0 \delta v_0 - \lambda \delta v) - H_2 \delta t, \\ \delta(\mu, z, v) &= \Sigma . m (\mu_0 \delta \tau_0 - \mu \delta \tau + z \delta \omega - z_0 \delta \omega_0 + v \delta \lambda - v_0 \delta \lambda_0) - H_2 \delta t, \\ \delta(\tau, \omega, \lambda) &= \Sigma . m (\tau \delta \mu - \tau_0 \delta \mu_0 + \omega_0 \delta z_0 - \omega \delta z + \lambda_0 \delta v_0 - \lambda \delta v) - H_2 \delta t, \\ \delta(\tau, \omega, v) &= \Sigma . m (\tau \delta \mu - \tau_0 \delta \mu_0 + \omega_0 \delta z_0 - \omega \delta z + v \delta \lambda - v_0 \delta \lambda_0) - H_2 \delta t, \\ \delta(\mu, z, \lambda) &= \Sigma . m (\mu_0 \delta \tau_0 - \mu \delta \tau + z \delta \omega - z_0 \delta \omega_0 + \lambda_0 \delta v_0 - \lambda \delta v) - H_2 \delta t, \\ \delta(\tau, z, \lambda) &= \Sigma . m (\tau \delta \mu - \tau_0 \delta \mu_0 + z \delta \omega - z_0 \delta \omega_0 + \lambda_0 \delta v_0 - \lambda \delta v) - H_2 \delta t, \\ \delta(\mu, \omega, v) &= \Sigma . m (\mu_0 \delta \tau_0 - \mu \delta \tau + \omega_0 \delta z_0 - \omega \delta z + v \delta \lambda - v_0 \delta \lambda_0) - H_2 \delta t, \end{aligned} \quad \left. \right\} (Y^2.)$$

$z_0 \lambda_0 \mu_0 v_0 \tau_0 \omega_0$ being the initial values of the varying elements $z \lambda \mu v \tau \omega$. If, then, we consider, for example, the first of these 8 combinations (τ, z, v) , as a function of

all the $3n - 3$ elements $\mu_i \omega_i \lambda_i$ and of their initial values $\mu_{0,i} \omega_{0,i} \lambda_{0,i}$ involving also in general the time explicitly, we shall have the following forms for the $6n - 6$ rigorous integrals of the $6n - 6$ equations (S².) :

$$\left. \begin{array}{l} m_i \tau_i = \frac{\delta}{\delta \mu_i} (\tau, z, v); \quad m_i \tau_{0,i} = - \frac{\delta}{\delta \mu_{0,i}} (\tau, z, v); \\ m_i z_i = \frac{\delta}{\delta \omega_i} (\tau, z, v); \quad m_i z_{0,i} = - \frac{\delta}{\delta \omega_{0,i}} (\tau, z, v); \\ m_i v_i = \frac{\delta}{\delta \lambda_i} (\tau, z, v); \quad m_i v_{0,i} = - \frac{\delta}{\delta \lambda_{0,i}} (\tau, z, v); \end{array} \right\} \dots \dots \dots \quad (Z^2.)$$

and in like manner we can deduce forms for the same rigorous integrals, from any one of the eight combinations (Y².) . The determination of all the varying elements would therefore be fully accomplished, if we could find the complete expression for any one of these 8 combinations.

40. A first approximate expression for any one of them can be found from the form under which we have supposed H_2 to be put, namely, as a function of the elements and of the time, which may be thus denoted :

$$H_2 = H_2 (\tau, z_1, \lambda_1, \mu_1, v_1, \tau_1, \omega_1, \dots z_{n-1}, \lambda_{n-1}, \mu_{n-1}, v_{n-1}, \tau_{n-1}, \omega_{n-1}); \quad (A^3.)$$

by changing in this function the varying elements to their initial values, and employing the following approximate integrals of the equations (S².) ,

$$\left. \begin{array}{l} \mu = \mu_0 + \frac{1}{m} \int_0^t \frac{\delta H_2}{\delta \tau_0} dt, \quad \tau = \tau_0 - \frac{1}{m} \int_0^t \frac{\delta H_2}{\delta \mu_0} dt, \\ \omega = \omega_0 + \frac{1}{m} \int_0^t \frac{\delta H_2}{\delta z_0} dt, \quad z = z_0 - \frac{1}{m} \int_0^t \frac{\delta H_2}{\delta \omega_0} dt, \\ \lambda = \lambda_0 + \frac{1}{m} \int_0^t \frac{\delta H_2}{\delta v_0} dt, \quad v = v_0 - \frac{1}{m} \int_0^t \frac{\delta H_2}{\delta \lambda_0} dt. \end{array} \right\} \dots \dots \dots \quad (B^3.)$$

For if we denote, for example, the first of the 8 combinations (Y².) by G, so that

$$G = \{\tau, z, v\}, \quad \dots \dots \dots \quad (C^3.)$$

we shall have, as a first approximate value,

$$G_1 = \int_0^t \left\{ \Sigma \left(\tau_0 \frac{\delta H_2}{\delta \tau_0} + z_0 \frac{\delta H_2}{\delta z_0} + v_0 \frac{\delta H_2}{\delta v_0} \right) - H_2 \right\} dt; \quad \dots \dots \dots \quad (D^3.)$$

and after thus expressing G_1 as a function of the time, and of the initial elements, we can eliminate the initial quantities of the forms $\tau_0 z_0 v_0$, and introduce in their stead the final quantities $\mu \omega \lambda$, so as to obtain an expression for G_1 of the kind supposed in (Z².) , namely, a function of the time t , the varying elements $\mu \omega \lambda$, and their initial values $\mu_0 \omega_0 \lambda_0$. An approximate expression thus found may be corrected by a process of that kind, which has often been employed in this Essay for other similar purposes. For the function G, or the combination (τ, z, v) , must satisfy rigorously, by (Y².) (A³.) , the following partial differential equation :

$$0 = \frac{\delta G}{\delta t} + H_2 \left(t, \frac{1}{m_1} \frac{\delta G}{\delta \omega_1}, \lambda_1, \mu_1, \frac{1}{m_1} \frac{\delta G}{\delta \lambda_1}, \frac{1}{m_1} \frac{\delta G}{\delta \mu_1}, \omega_1, \frac{1}{m_2} \frac{\delta G}{\delta \omega_2}, \dots, \omega_{n-1} \right); \dots \quad (E^3)$$

and each of the other analogous functions or combinations (Y^2 .) must satisfy an analogous equation: if then we change G to $G_1 + G_2$, and neglect the squares and products of the coefficients of the small correction G_2 , G_1 being a first approximation such as that already found, we are conducted, as a second approximation, on principles already explained, to the following expression for this correction G_2 :

$$G_2 = - \int_0^t \left\{ \frac{\delta G_1}{\delta t} + H_2 \left(t, \frac{1}{m_1} \frac{\delta G_1}{\delta \omega_1}, \lambda_1, \mu_1, \frac{1}{m_1} \frac{\delta G_1}{\delta \lambda_1}, \frac{1}{m_1} \frac{\delta G_1}{\delta \mu_1}, \omega_1, \dots \right) \right\} dt: \quad (E^3)$$

which may be continually and indefinitely improved by a repetition of the same process of correction. We may therefore, theoretically, consider the problem as solved; but it must remain for future consideration, and perhaps for actual trial, to determine which of all these various processes of successive and indefinite approximation, deduced in the present Essay and in the former, as corollaries of one general Method, and as consequences of one central Idea, is best adapted for numeric application, and for the mathematical study of phenomena.



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